Automata Networks

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Boolean Automata Networks: discrete deterministic dynamical systems over a graph only with states 0 or 1 on the vertices.

A vertex updates it state following a boolean rule depending of the states of its graph's neighbors.

The dynamics

Parallel or synchronous update: Every site is updated at the same time.

Sequential update: sites are updated one by one in a prescribed order.

Block- sequential updates:

Consider a partition $\{I_1,\ldots,I_p\}$ of the set {1, …, n}

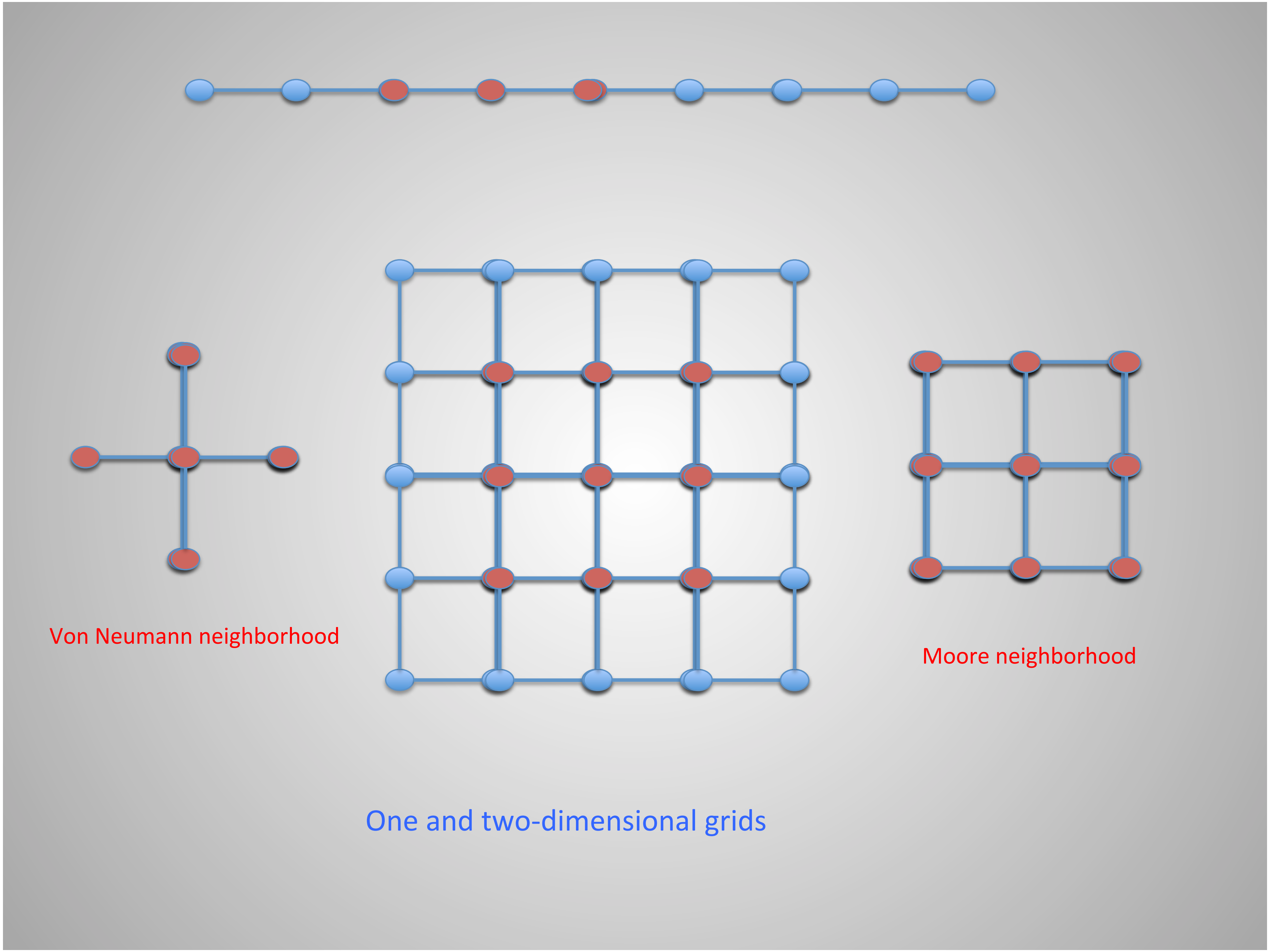
We update the blocks one by one:

To update the k-th block we consider the new state of every sites belong to previous blocks.

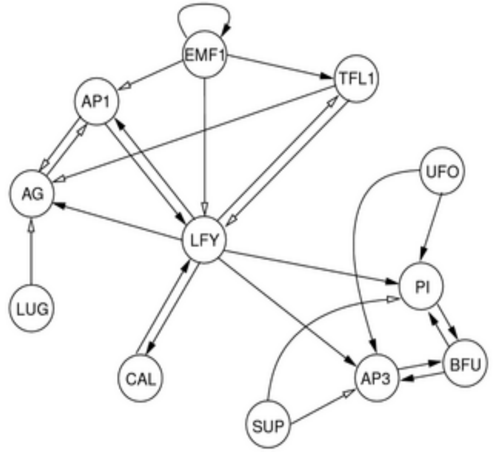
Particular cases

Cellular Automata: on d-dimensional grids
with the same local function in every site,
updated (usually) in parallel.

Arbitrary finite graph modelling regulatory genetic networks


Von Neumann neighborhood
Moore neighborhood
One and two-dimensional grids

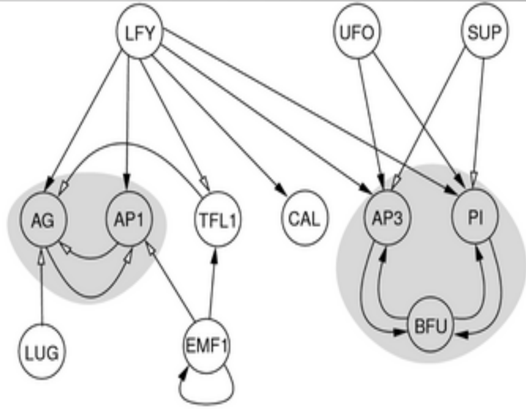
Arabidopsis regulation threshold network



$W =$

	EMF1	TFL1	LFY	AP1	CAL	LUG	UFO	BFU	AG	AP3	PI	SUP
EMF1	1	0	0	0	0	0	0	0	0	0	0	0
TFL1	1	0	−2	0	0	0	0	0	0	0	0	0
LFY	−2	−1	0	2	1	0	0	0	0	0	0	0
AP1	−1	0	5	0	0	0	0	0	−1	0	0	0
CAL	0	0	2	0	0	0	0	0	0	0	0	0
LUG	0	0	0	0	0	0	0	0	0	0	0	0
UFO	0	0	0	0	0	0	0	0	0	0	0	0
BFU	0	0	0	0	0	0	0	0	0	1	1	0
AG	0	−2	1	−2	0	−1	0	0	0	0	0	0
AP3	0	0	3	0	0	0	2	1	0	0	0	−2
PI	0	0	4	0	0	0	1	1	0	0	0	−1
SUP	0	0	0	0	0	0	0	0	0	0	0	0

$$\Theta = \begin{pmatrix} 0 \\ 0 \\ 3 \\ -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$



$\widetilde{W} =$

	EMF1	TFL1	LFY	AP1	CAL	LUG	UFO	BFU	AG	AP3	PI	SUP
EMF1	1	0	0	0	0	0	0	0	0	0	0	0
TFL1	1	0	−2	0	0	0	0	0	0	0	0	0
LFY	0	0	0	0	0	0	0	0	0	0	0	0
AP1	−2	0	5	0	0	0	0	0	−2	0	0	0
CAL	0	0	2	0	0	0	0	0	0	0	0	0
LUG	0	0	0	0	0	0	0	0	0	0	0	0
UFO	0	0	0	0	0	0	0	0	0	0	0	0
BFU	0	0	0	0	0	0	0	0	0	1	1	0
AG	0	−2	1	−2	0	−1	0	0	0	0	0	0
AP3	0	0	3	0	0	0	2	1	0	0	0	−2
PI	0	0	4	0	0	0	1	1	0	0	0	−1
SUP	0	0	0	0	0	0	0	0	0	0	0	0

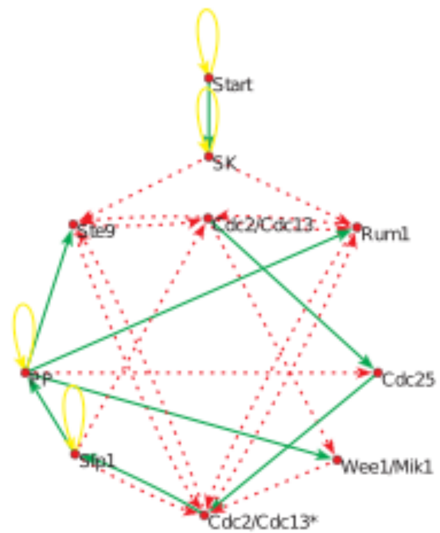
$$\tilde{\Theta} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Bioinformatics. 1999 Jul-Aug;15(7-8):593-606.
Genetic control of flower morphogenesis in Arabidopsis thaliana: a logical analysis. Mendoza L, Thieffry D, Alvarez-Buylla ER.

Demongeot J, G. E, Morvan M, Noual M, Sené S (2010) Attraction Basins as Gauges of Robustness against Boundary Conditions in Biological Complex Systems. PLoS ONE 5(8): e11793. doi:10.1371

Yeast cell-cycle Threshold Networks

Fig. 2 The fission yeast cell-cycle threshold Boolean network. Using the same configuration as (Davidich and Bornholdt 2008), the *green/solid edges* represent positive weights (activations), the *red/dashed edges* represent negative weights (inhibitory). The *yellow/solid loops* represent self-degradation



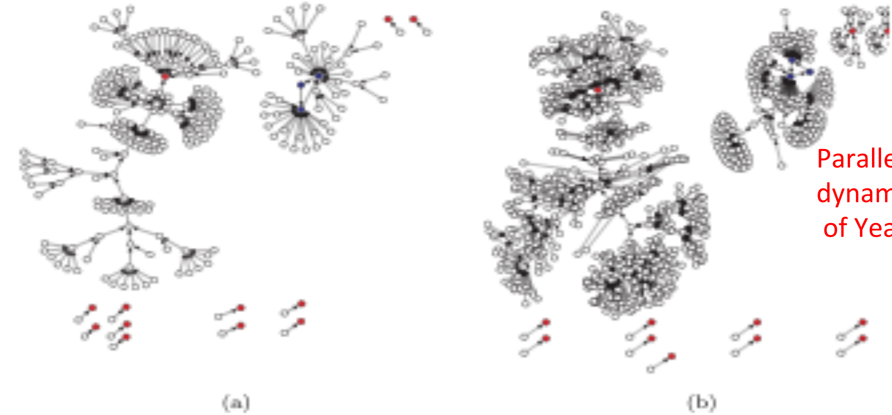
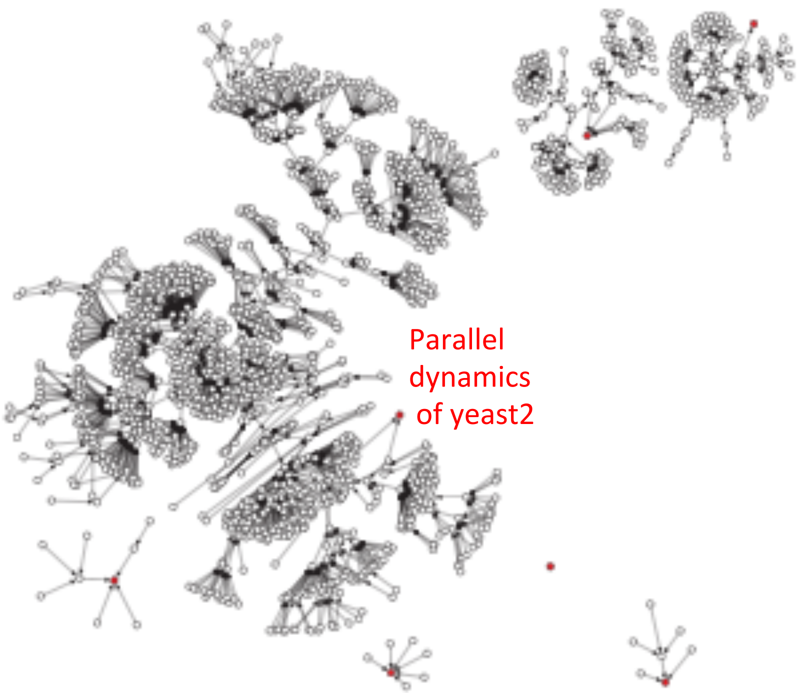


Fig. 7 (**a**) State transition graph for *Yeast1*, without *Start* and *SK*, using the parallel updating schen (**b**) State transition graph for the complete *Yeast1*, using the parallel updating scheme. The *twelve* circles represent the fixed point states, the *three blue circles* represent the states that belong to the lii



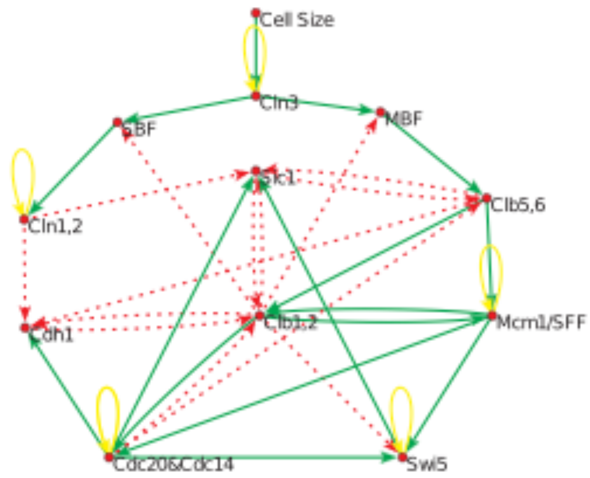
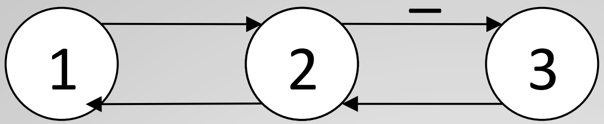


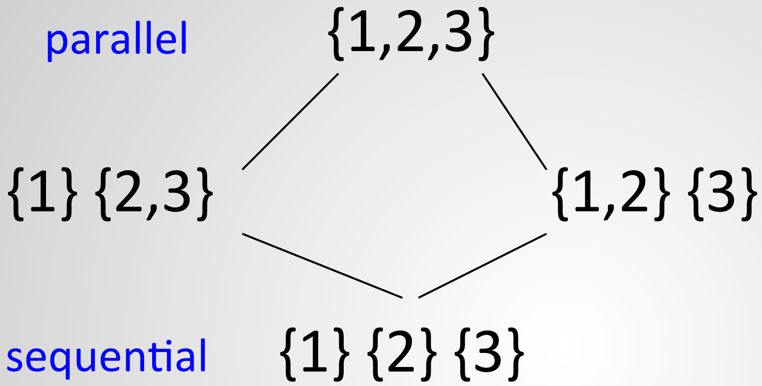
Fig. 10 The budding yeast cell-cycle threshold Boolean network. Using the same configuration as (Li et al. 2004), the *green/solid edges* represent positive weights (activations), the *red/dashed edges* represent negative weights (inhibitory). The *yellow/solid loops* represent self-degradation

Decontruction and Dynamical robustness of regulatory networks: application to the Yeast cell cycle networks. E.G, M. Montalva and G. Ruz, Bull Math Biol (2013) 75, 939-966

The(directed) graph



EXAMPLE



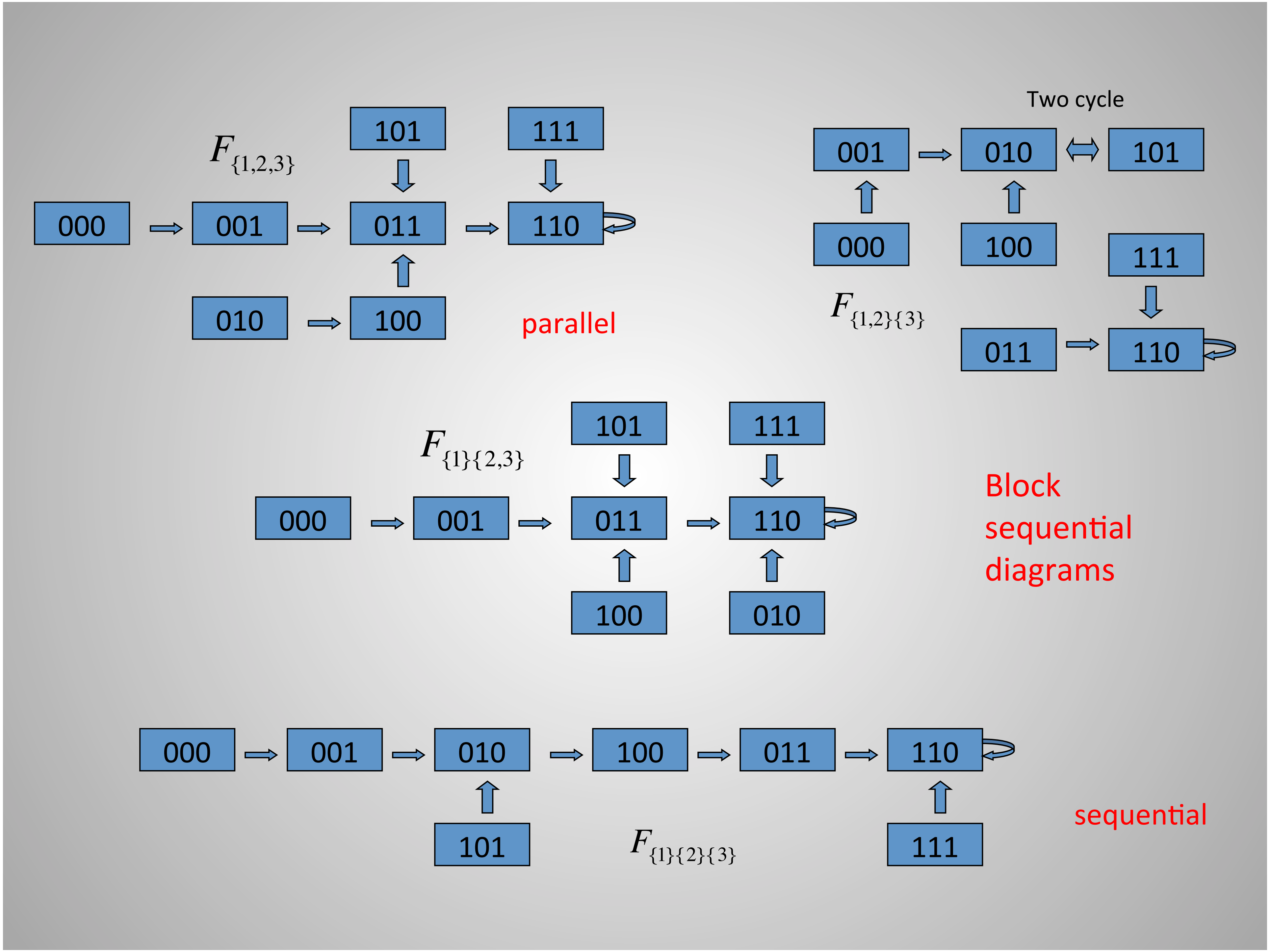
Some Block-Sequential partitions for three sites

$$F_{\{1,2,3\}}(x_1,x_2,x_3) = (x_2, x_1 + x_3, \neg x_2)$$

$$F_{\{1,2\}\{3\}}(x_1,x_2,x_3) = (x_2, x_1 + x_3, (\neg x_1)(\neg x_3))$$

$$F_{\{1\}\{2,3\}}(x_1,x_2,x_3) = (x_2, x_2 + x_3, \neg x_2)$$

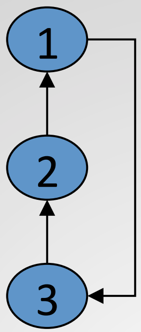
$$F_{\{1\}\{2,3\}}(x_1,x_2,x_3) = (x_2, x_2 + x_3, (\neg x_2)(\neg x_3))$$



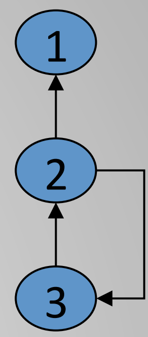
$F_{\{1,2,3\}}$
101
111
000
001
011
110
010
100
parallel
Two cycle
001
010
101
000
100
111
$F_{\{1,2\}\{3\}}$
011
110
$F_{\{1\}\{2,3\}}$
101
111
000
001
011
110
100
010
Block sequential diagrams
000
001
010
100
011
110
101
$F_{\{1\}\{2\}\{3\}}$
111
sequential

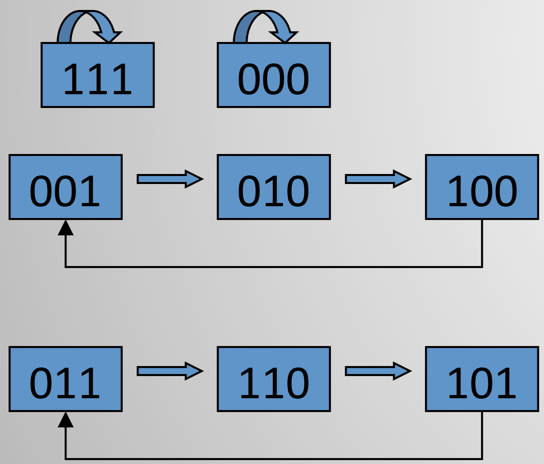
Cycles for synchronous and sequential updates

$$F:\{0,1\}^3 \to \{0,1\}^3$$
$$f_1(x_1,x_2,x_3) = x_2$$
$$f_2(x_1,x_2,x_3) = x_3$$
$$f_3(x_1,x_2,x_3) = x_1$$

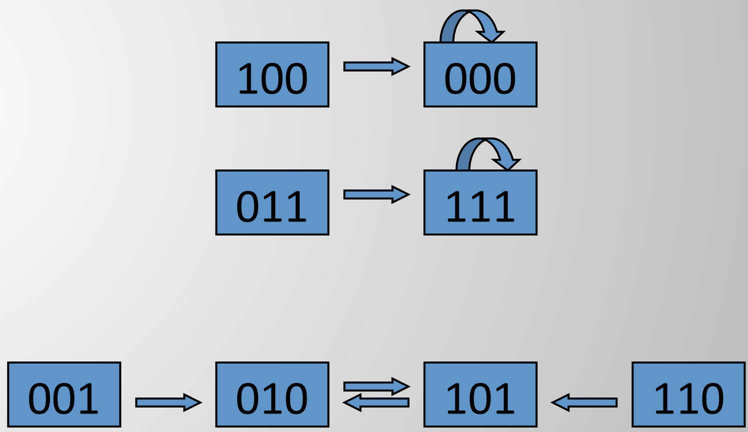


$$G:\{0,1\}^3 \to \{0,1\}^3$$
$$g_1(x_1,x_2,x_3) = x_2$$
$$g_2(x_1,x_2,x_3) = x_3$$
$$g_3(x_1,x_2,x_3) = x_2$$





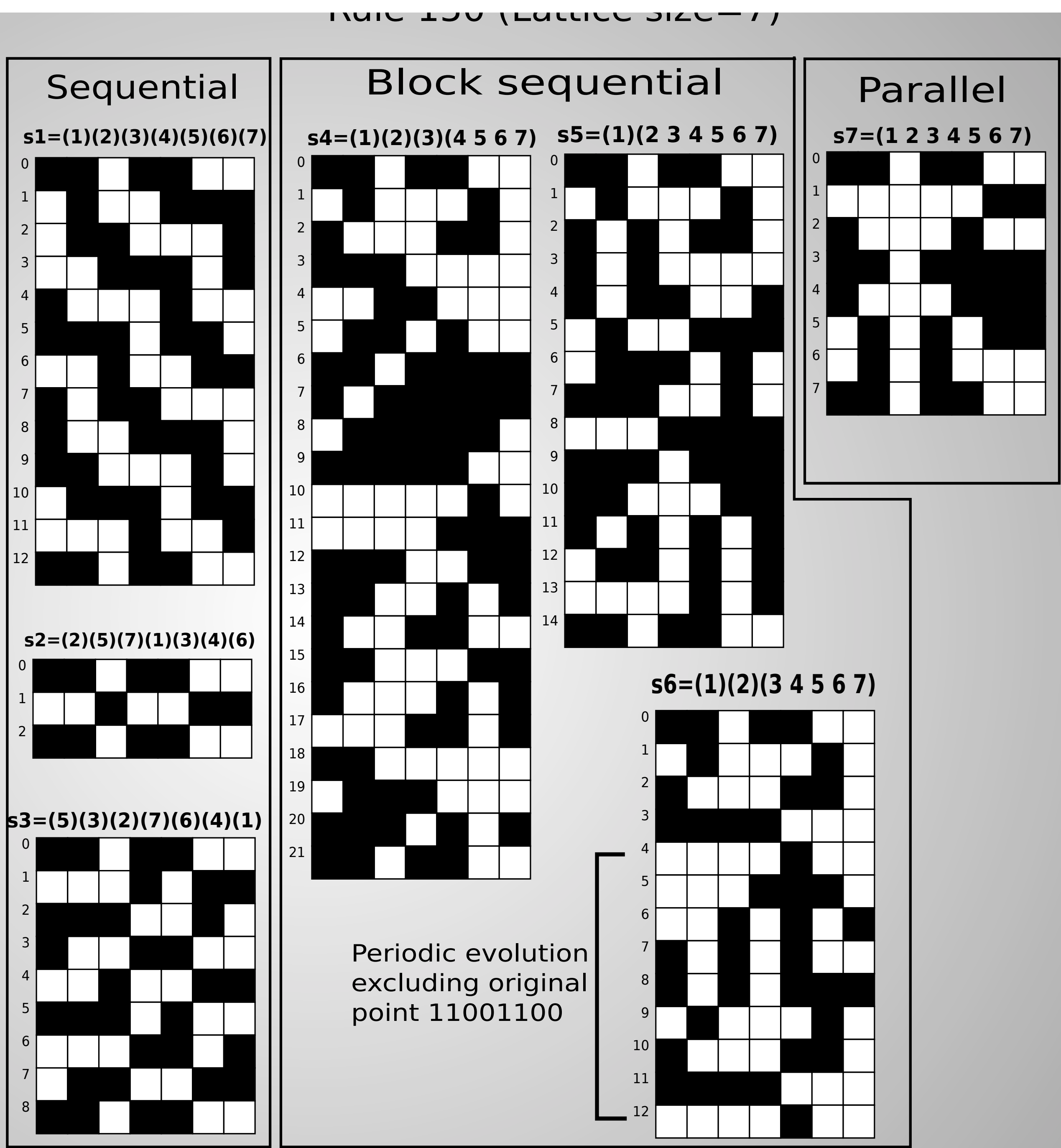
Parallel update: 3-cycles



Sequential update: 2-cycle

Elementary one dimensional Automaton with periodic conditions and the linear local rule 150

000 0
001 1
010 1
011 0
100 1
101 0
110 0
111 1



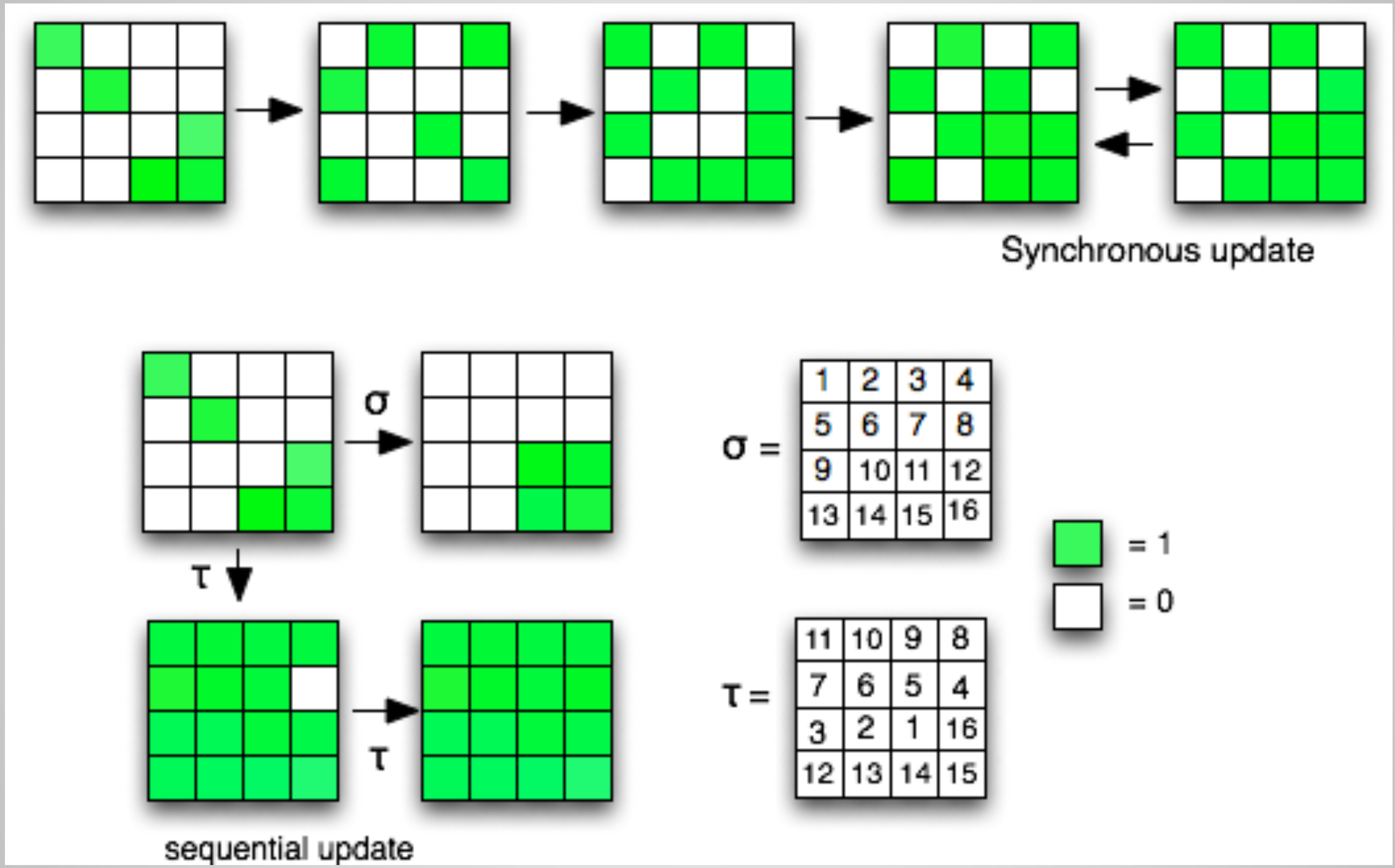
dynamics for the majority rule in a two dimensional grid

We consider a 4x4 lattice with periodic conditions,
nearest interactions, states 0 or 1, and the local majority function:

If the number of ones is bigger or equal to the number of zeros then the site takes the value 1

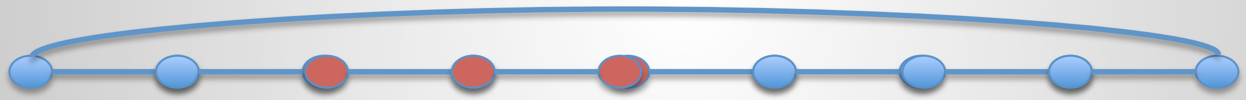
$$x'_{ij} = 1 \quad iff \quad x_{i-1,j} + x_{i+1,j} + x_{i,j-1} + x_{i,j+1} \geq 2$$

Dynamics: two cycles and fixed points; different behavior for different updates



Different updates give different dynamic behavior

Different updates schemes on one dimensional elementary periodic CAs



n≥3 sites

An ECA is sequential invariant If and only if for n>3   and any permutation (sequential update), the set of attractors is the same.

Clearly, if the ECA admits only fixed points it is trivially sequential invariant

Theorem (Macauley, Mcammond,Morveit (JCA2013)

There exists 41 non-equivalent sequential-invariant ECA

Rule	$f_i(u,v,w)$	Rule	$f_i(u,v,w)$
0	0	108	$u\overline{v}w + v\overline{w} + \overline{u}v$
1	$\overline{u}\,\overline{v}\,\overline{w}$	128	uvw
4	$\overline{u}v\overline{w}$	129	$\overline{u}\,\overline{v}\,\overline{w} + uvw$
5	$\overline{u}\,\overline{w}$	132	$\overline{u}v\overline{w} + uvw$
8	$\overline{u}vw$	133	$uvw + \overline{u}\,\overline{w}$
9	$\overline{u}\,\overline{v}\,\overline{w} + \overline{u}vw$	136	vw
12	$\overline{u}v$	137	$\overline{u}\,\overline{v}\,\overline{w} + vw$
13	$\overline{u}\,\overline{w} + \overline{u}v$	140	$\overline{u}v + vw$
28	$u\overline{v}\,\overline{w} + \overline{u}v$	141	$\overline{u}\,\overline{w} + vw$
29	$\overline{v}\,\overline{w} + \overline{u}v$	150	$u\overline{v}\,\overline{w} + \overline{u}v\overline{w} + \overline{u}\,\overline{v}w + uvw$
32	$u\overline{v}w$	152	$u\overline{v}\,\overline{w} + vw$
40	$u\overline{v}w + \overline{u}vw$	156	$u\overline{v}\,\overline{w} + \overline{u}v + vw$
51	$\overline{v}$	160	uw
54	$\overline{u}v\overline{w} + u\overline{v} + \overline{v}w$	164	$\overline{u}v\overline{w} + uw$
57	$\overline{u}vw + \overline{v}\,\overline{w} + u\overline{v}$	168	$uw + vw$
60	$u\overline{v} + \overline{u}v$	172	$\overline{u}v + uw$
72	$uv\overline{w} + \overline{u}vw$	184	$u\overline{v} + vw$
73	$\overline{u}\,\overline{v}\,\overline{w} + uv\overline{w} + \overline{u}vw$	200	$uv + vw$
76	$v\overline{w} + \overline{u}v$	204	v
77	$\overline{u}\,\overline{w} + v\overline{w} + \overline{u}v$	232	$uv + uw + vw$
105	$\overline{u}\,\overline{v}\,\overline{w} + uv\overline{w} + u\overline{v}w + \overline{u}vw$		

TABLE 1
The 41 representative rules π-independent of [5] and its minimal disjunctive normal form representations. At each time step, the value x_i is updated according to the rule $x'_i = f_i(u,v,w)$ where $u = x_{i-1}$, $v = x_i$ and $w = x_{i+1}$.

Between those 41 rules we are interested
To characterize those which are invariant for every
Block Sequential update

For instance the rule 5 is not block-sequential invariant

Rule 5

It is enough to consider n=4

$f(0,0,0)=1$
$f(x,y,z)=0$ else

And the updates (3,4)(1,2) and (4)(3)(1,2)

The attractors for (3,4)(1,2) are: 0000; 0011; 1100; i.e., 0, 3, 12

3-cycle

0000
0011
1100
0000

The attractors for (4)(3)(1,2) are: 0000 and 0101, i.e. 0 and 5

0000
0101
0000

Theorem (Montalva, Morveit, Ramirez, EG)

For $n>3$, there are 15 non-equivalent rules

block-sequential invariants.

0, 4, 8, 12, 28, 51, 72, 76, 128, 132, 136, 140, 141, 200, 204

Theorem (Montalva, Morveit, Ramirez, EG)

For n>3, there are 15 non-equivalent rules

block-sequential invariants.

0, 4, 8, 12, 28, 51, 72, 76, 128, 132, 136, 140, 141, 200, 204

consider for instance the rule 32 which is not block-invariant
But it is for n odd.

000 0
001 0
010 0
011 0
100 0
101 1
110 0
111 0

Clearly * 00* remains stable. And at each step
0s are added at both sides …

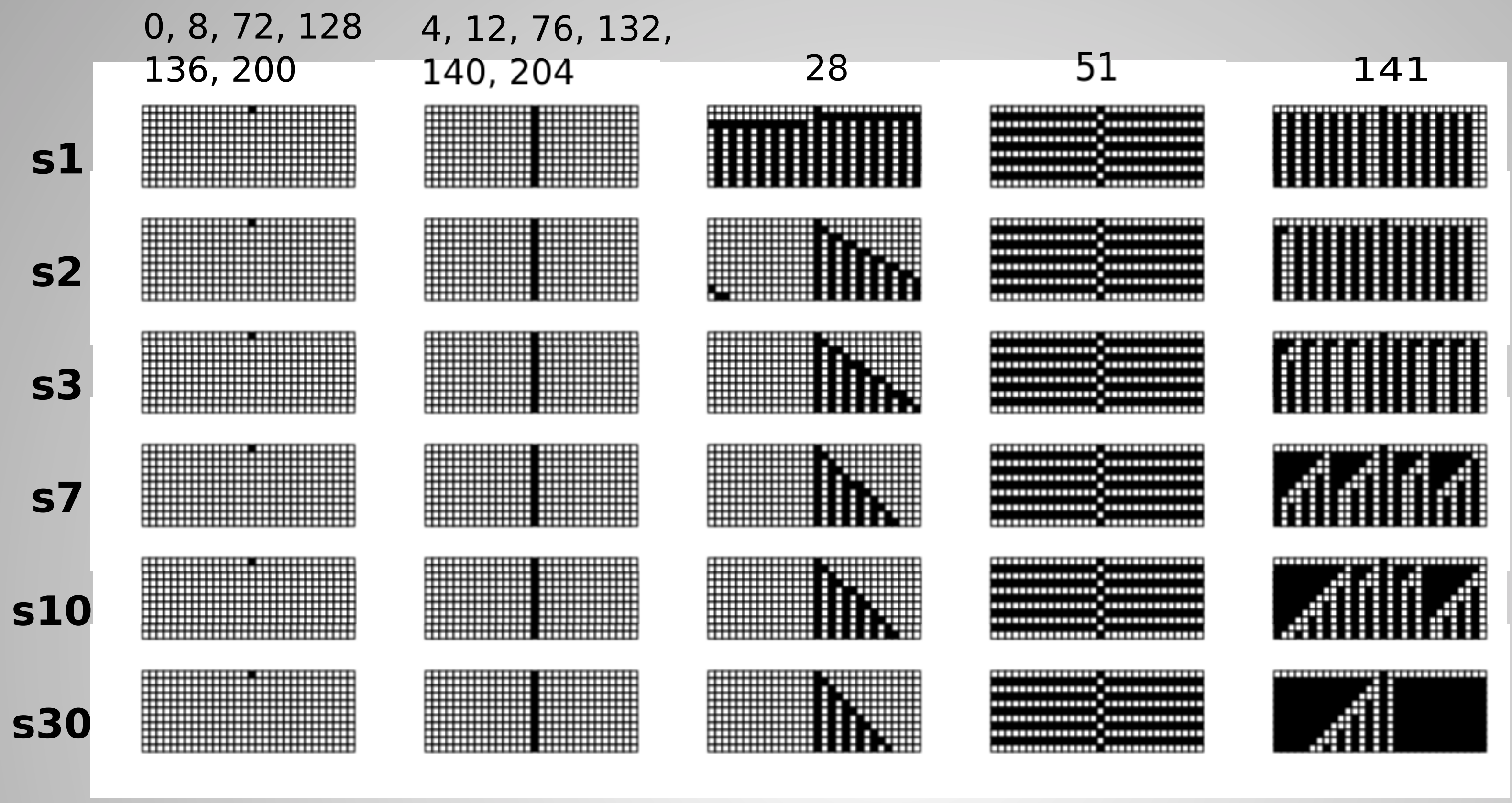
The other case: 10101010 …..
is a two cycle only for n even

If there are two 1's together every update create a block 00

So the rule 32 is block- invariant only for n odd

(9n parallel the 2-cycle exist only for n even).

Block invariant rules



Evolution of the 15 block invariant Wolfram's rules starting with a single one in the 16th position, under the block updates $s_1 = (1)(2)\cdots(30)$, $s_2 = (1,2)(3,4)\cdots(29,30)$, $s_3 = (1,2,3)(4,5,6)\cdots(28,29,30)$, $s_7 = (1,...,7)(8,...,14)\cdots(22,...,28)(29,30)$, $s_{10} = (1,...,10)\cdots(21,...,30)$ and $s_{30} = (1,...,30)$.

Threshold networks

$$x_i' = H\left(\sum_{j=1}^{n} w_{ij} x_j - b_i\right) \quad for \quad 1 \le i \le n \qquad x \in \{0,1\}^n$$

$W = (w_{ij})$ the weight integral matrix

$b = (b_i)$ the threshold vector

$H(u) = 1$ if $u \ge 0$

0 otherwise

E.G and G. Ruz, Dynamics of Neural Networks over undirected graphs, Neural Networks, Vol 63, 156-169, 2015

For arbitrary matrices W previous model may accept, Iterated in parallel or sequentially, long period cycles and transients …

But when W is symmetric the network converges to fixed point or two periodic cycles (parallel update),

And, if diag(W)≥0 to fixed point (sequential update).

E.G, J. Olivos, Periodic behaviour of generalized threshold functions, Discrete mathematics, vol 30, pp 187-189, 1980.

E.G., Fixed Point behavior of threshold functions on a finite set, SIAM Journal on Alg. And Discrete Methods, vol 3(4), pp 2554-2558, 1982.

Further for W symmetric the network admits an energy:

Parallel update:

$$E(x(t)) = -\sum_{i=1}^{n} x_i(t) \sum_{j=1}^{n} w_{ij} x_j(t-1) + \sum_{i=1}^{n} b_i (x_i(t) + x_i(t-1))$$

If diag (W) ≥ 0, Sequential update:

$$E(x) = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} x_i x_j + \sum_{i=1}^{n} b_i x_i$$

Which implies that:

1) for the parallel updating the attractors are only
Fixed points or two cycles.

2) For the sequential updating and diag(W)≥0 there are only fixed points.

3) In both situations transients are bounded by α||W||x||b||

$x' \neq x$

$$\Delta E = E(x(t)) - E(x(t-1) < 0$$ If and only if $x(t) \neq x(t-2)$

And for the sequential iteration

$$\Delta E = E(x') - E(x) < 0$$ iff $x' \neq x$

The most general dynamical result about Threshold networks:

Consider the block-sequential scheme $s = \{I_1, \ldots, I_p\}$

The symmetrical threshold network *T*=(W, b, *s*)

Let $W(I_k)$ the sub-matrix associated to the k-th block

If for every $k \in \{1, \ldots, p\}$ $W(I_k)$ is non-negative-definite

The network converges to fixed points

E. G., F. Fogelman-Soulie, D. Pellegrin, Decreasing energy functions as a tool
For studying threshold networks, Discrete Applied Mathematics, vol 12, pp261-277, 1985.

Sketch of the proof:

The update of the k-th block: $x' = (x_{I_1}, \ldots, x_{I_{k-1}}, x'_{I_k}, x_{I_{k+1}}, \ldots, x_{I_p})$

$$\Delta E = -\sum_{i \in I_k} (x'_i - x_i)\left(\sum_{j=1}^{n} w_{ij} x_j - b_i\right) - \frac{1}{2} \sum_{i \in I_k} (x'_i - x_i) \sum_{i \in I_k} (x_j' - x_j)$$

$$\Delta E = \sum_{i \in I_k} \delta_i - \frac{1}{2} y^t W(I_k) y \quad -\frac{1}{2} y^t W(I)$$ where $y = (x' - x) \in \{-1, 0, 1\}^n$

$$\delta_i = -(x'_i - x_i)\left(\sum_{j=1}^{n} w_{ij} x_j - b_i\right)$$

Since W(I) is non-negative definite $-\frac{1}{2} y^t W y \leq 0$

and x’≠x $\Rightarrow$ there exists $i \in \{1, .., n\}$ such that $\delta_i \leq -\frac{1}{2}$

(since W is an integral matrix)

Then $\Delta E < 0$

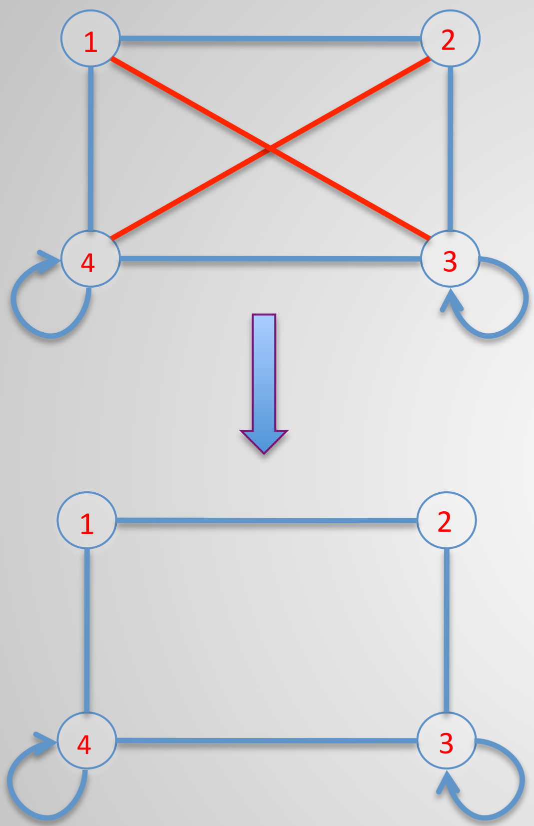
We will suppose now that every matrix W is the incidence matrix of an undirected graph G=(V,E), so their entries belong to the set {0,1}
W=W(G)= (w_{ij}) eventually with loops $(w_{ii} = 1)$

Consider the quantity:

$$\alpha(G) = -n - k + 2m - 4p$$

n = |V|,
m =|E|, (without loops)
K = the number of loops,
p = the minimum number of edges to remove
such that the sub-graph is bipartite.

Example



$|V| = 4$ $k = 2$

$|E| = 6$ $p = 2$

Maximum bipartite sub-graph

$$\alpha(G) = -4 - 2 + 2 \times 6 - 4 \times 2 = -2 < 0$$

Theorem-1

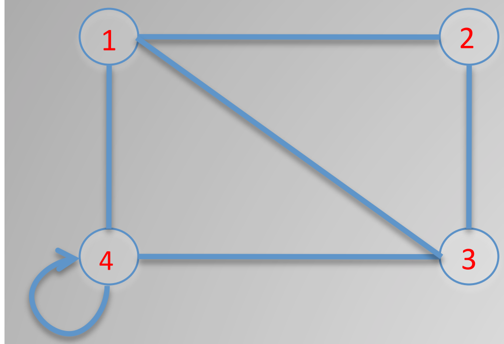
Consider an undirected graph G=(V,E), W=W(G), b a threshold vector.

and the network updated in parallel, N= (W, b, {1, …,n})

$\alpha(G') < 0$ For any G’ sub-graph of G (by deleting vertices) $\Rightarrow$ Fixed points for any threshold vector

$\alpha(G') \geq 0 \Rightarrow$ There exists a threshold vector such that two cycles appears

In E.G and G. Ruz, Dynamics of Neural Networks over undirected graphs, Neural Networks, Vol 63, 156-169, 2015



$$f_1(x) = H(x_2 + x_1 + x_4 - \frac{3}{2})$$

$$f_2(x) = H(x_1 + x_2 + x_3 - \frac{3}{2})$$

$$f_3(x) = H(x_2 + x_3 + x_4 - \frac{3}{2})$$

$$f_4(x) = H(x_1 + x_4 + x_3 - \frac{3}{2})$$

$\Rightarrow$

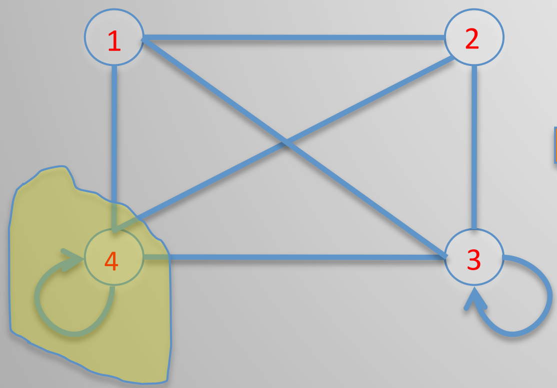
Two-cycle

$$(x_1, x_2, x_3, x_4) = (1,0,1,0) \leftrightarrow (0,1,0,1)$$

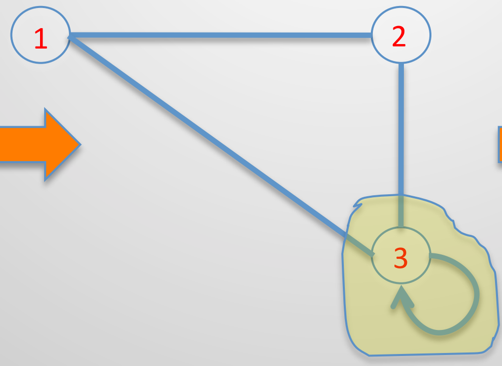
$\alpha(G) = -4 - 1 + 2 \times 5 - 4 = 1 \geq 0$ So there exists a two-cycle.

There exists a strict sub-graph with $\alpha(G) \geq 0$

$\alpha(G) = -2$



$\alpha(G) = -2$



$\alpha(G) = 0$

$$f_1(x) = H(x_2 - \frac{1}{2})$$

$$f_2(x) = H(x_1 - \frac{1}{2})$$

$$(1,0) \leftrightarrow (0,1)$$

Two-cycle

Parallel updating on two families of graphs

Bipartite graphs (k=0)
with n loops
(diag (W)=(1,…1))

$$\alpha(G) = -2n + 2m$$

If G is a forest) then n>m (with loops) $\Rightarrow$ Only fixed points

Complete graphs with n loops

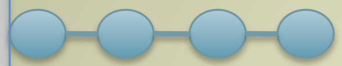
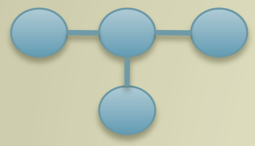
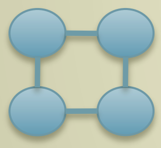
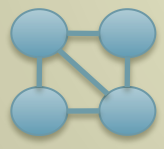
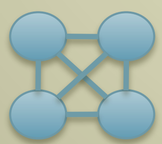
In this situation, the minimum number of edges to remove to obtain a bipartite graph

$p = 2q(q-1)$ for n=2q

$p = 2q^2$ for n=2q+1

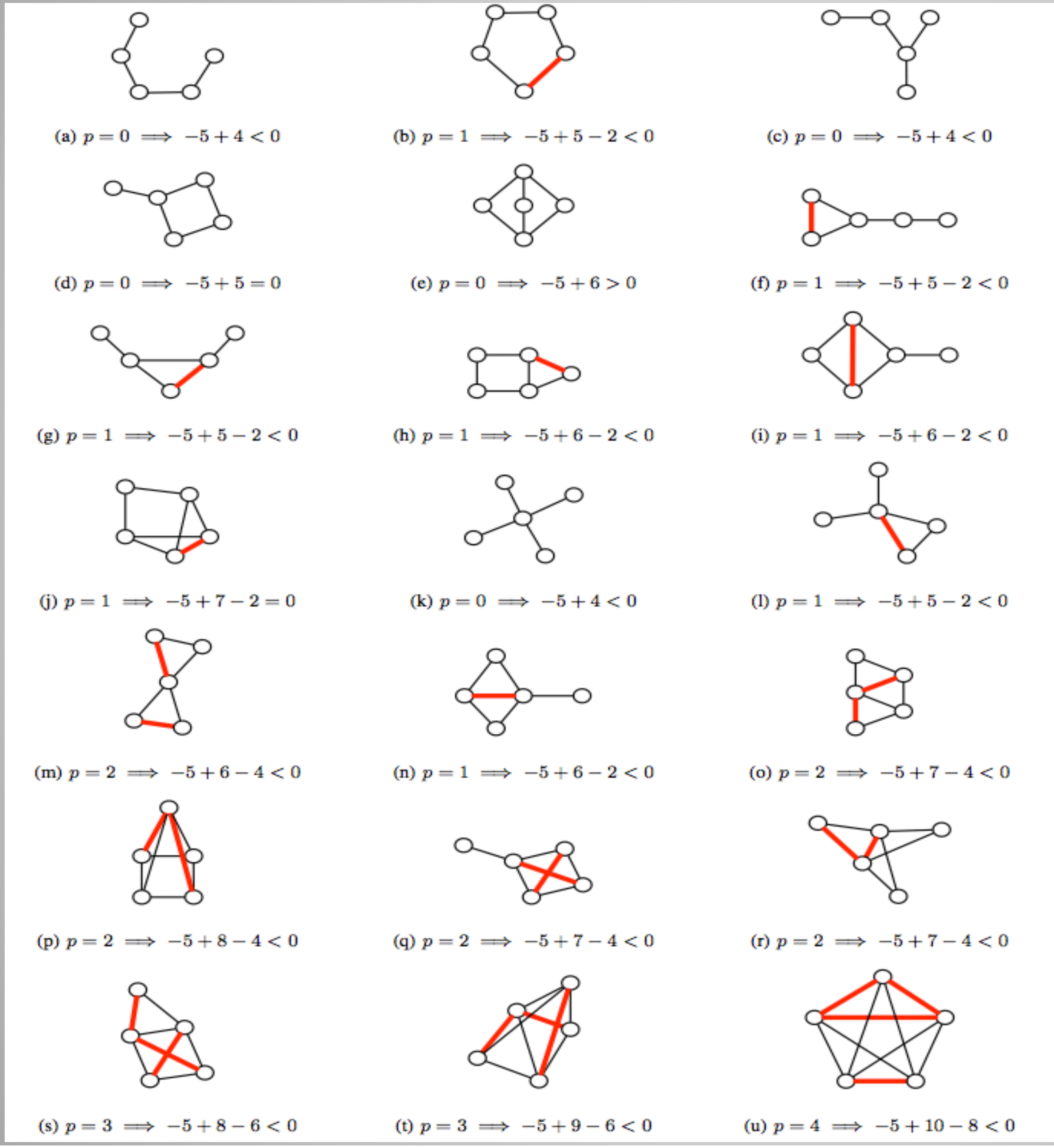
$\alpha(K_n) < 0 \Rightarrow$ Complete graphs updated in Parallel converges to fixed points

Parallel Updating

n=4	Fixed points	Two-Cycles
	$3 \leq k \leq 4$	$0 \leq k \leq 2$
	$1 \leq k \leq 4$	$k=0$
	$\varnothing$	$0 \leq k \leq 4$
	$3 \leq k \leq 4$	$0 \leq k \leq 2$
	$1 \leq k \leq 4$	$k=0$

k=number of loops

Connected graphs for n=5 with 5 loops.



(a) $p = 0 \implies -5 + 4 < 0$

(b) $p = 1 \implies -5 + 5 - 2 < 0$

(c) $p = 0 \implies -5 + 4 < 0$

(d) $p = 0 \implies -5 + 5 = 0$

(e) $p = 0 \implies -5 + 6 > 0$

(f) $p = 1 \implies -5 + 5 - 2 < 0$

(g) $p = 1 \implies -5 + 5 - 2 < 0$

(h) $p = 1 \implies -5 + 6 - 2 < 0$

(i) $p = 1 \implies -5 + 6 - 2 < 0$

(j) $p = 1 \implies -5 + 7 - 2 = 0$

(k) $p = 0 \implies -5 + 4 < 0$

(l) $p = 1 \implies -5 + 5 - 2 < 0$

(m) $p = 2 \implies -5 + 6 - 4 < 0$

(n) $p = 1 \implies -5 + 6 - 2 < 0$

(o) $p = 2 \implies -5 + 7 - 4 < 0$

(p) $p = 2 \implies -5 + 8 - 4 < 0$

(q) $p = 2 \implies -5 + 7 - 4 < 0$

(r) $p = 2 \implies -5 + 7 - 4 < 0$

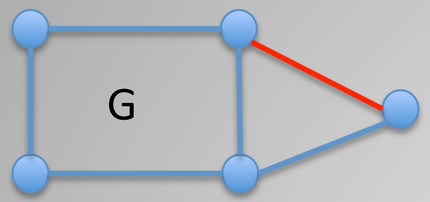
(s) $p = 3 \implies -5 + 8 - 6 < 0$

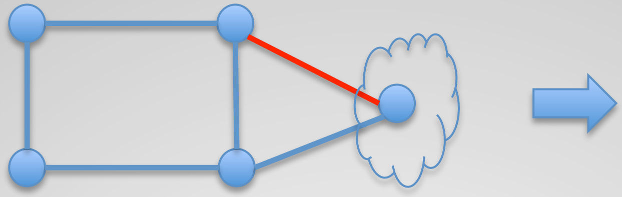
(t) $p = 3 \implies -5 + 9 - 6 < 0$

(u) $p = 4 \implies -5 + 10 - 8 < 0$

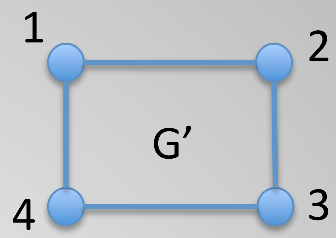
$$\frac{\alpha(G)}{2} = -n + m - 2p$$

In red the edges to be removed for a maximum bipartite graphs





(delete)



$\alpha(G) = -5 - 5 + 2x6 - 4 = -2$

$\alpha(G) = -5 - 5 + 2x6 - 4 = -2$

(Each vertex has a loop)

(1,0,1,0) ⟺ (0,1,1,0)

Two-cycle

$$f_1(x) = H(x_2 + x_1 + x_4 - \frac{3}{2})$$

$$f_2(x) = H(x_1 + x_2 + x_3 - \frac{3}{2})$$

$$f_3(x) = H(x_2 + x_3 + x_4 - \frac{3}{2})$$

$$f_4(x) = H(x_1 + x_4 + x_3 - \frac{3}{2})$$

Theorem-II: attractors for every block-sequential update.

Consider the block-sequential scheme $s = \{I_1, \ldots, I_p\}$

The symmetrical threshold network T=(W, b, s)

Let $G(I_k)$ the graph associated to the k-th block

$\forall\, k \in \{1, \ldots, p\}\ \ \alpha(G') < 0\ \ \forall\, G' \subseteq G(I_k) \Rightarrow$ fixed points

$\exists\, k \in \{1, \ldots, p\}$ and $G' \subseteq G(I_k)$ such that $\alpha(G') \geq 0 \Rightarrow$ cycles

Corollary

No more than three sites at each block

Consider an undirected graph G=(V,E) with every loop and the

the block-sequential scheme $s = \{I_1, \ldots, I_p\}$

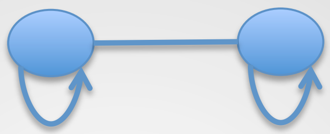
$\forall\, k \in \{1, \ldots, p\} \ \ |I_k| \leq 3 \ \Rightarrow$ Fixed points

Otherwise, there exists graphs and threshold vectors such that cycles appear

Sketch of the proof:

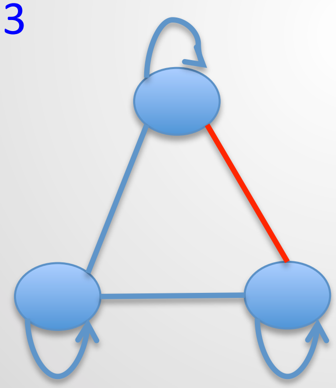
Partition size =1 directly from the fact that diag(W)≥0

Partition size = 2

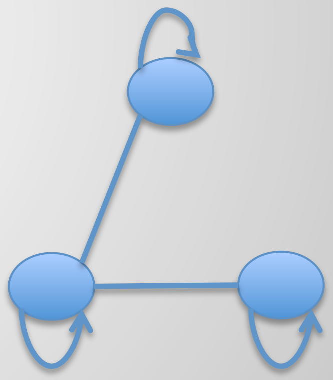


$\alpha(G) = -2$

Partition size= 3

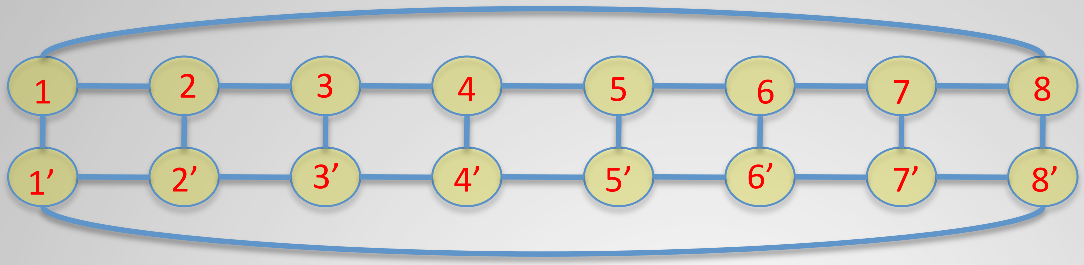


$\alpha(G) = -4$

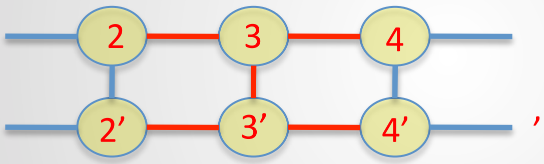


$\alpha(G) = -2$

Non-Polynomial Cycles



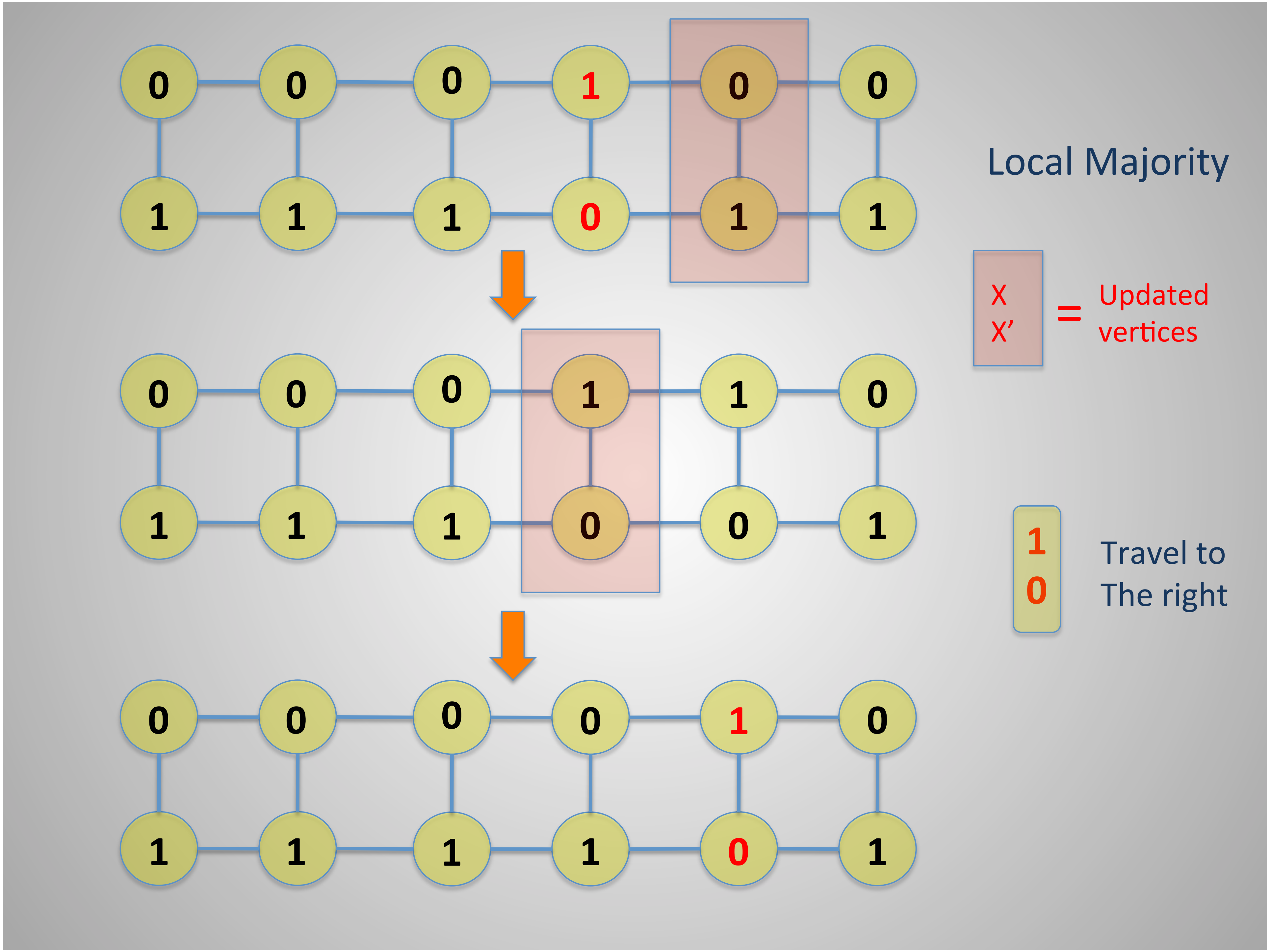
staircase



Local majority at each vertex

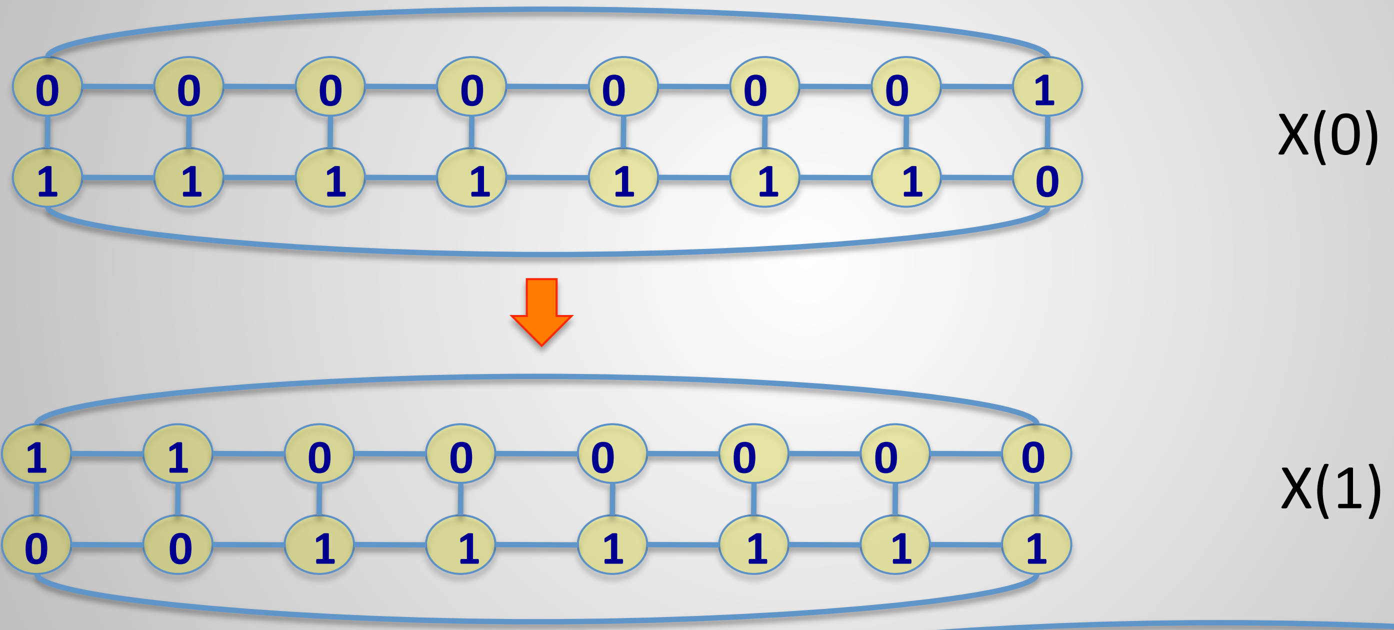
$$f_3(x) = H(x_2 + x_{3'} + x_4 - \frac{3}{2})$$

$$f_{3'}(x) = H(x_{2'} + x_3 + x_{4'} - \frac{3}{2})$$

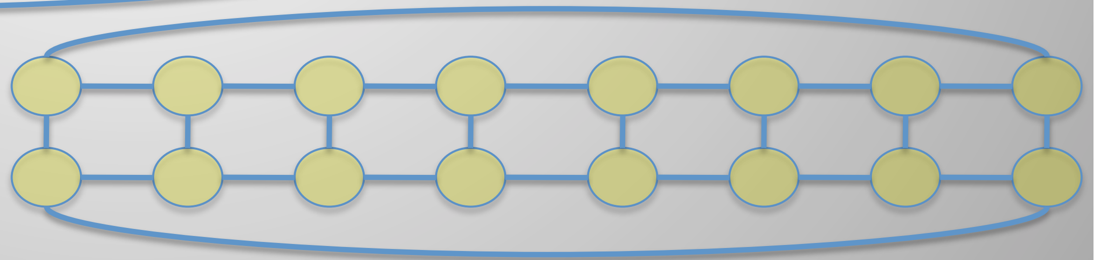

0 0 0 1 0 0
1 1 1 0 1 1
0 0 0 1 1 0
1 1 1 0 0 1
0 0 0 0 1 0
1 1 1 1 0 1
Local Majority
X
X’
= Updated vertices
1
0
Travel to
The right

Block-Sequential updating

$$\tau = \{\{1,1'\},\{n,n'\},\{n-1,(n-1)'\},\ldots\{3,3'\},\{2,2'\}\}$$

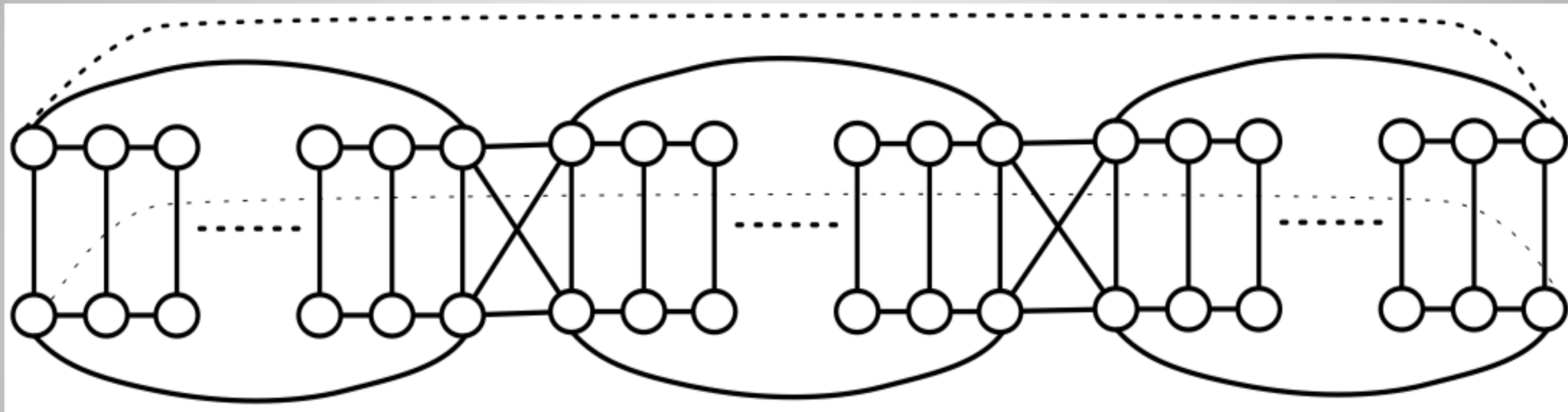


Cycle of period T=n-1



Union of the first l prime number's staircases of size

$$p_1 + 1 = 3; p_2 + 1 = 4; p_3 + 1 = 6, p_4 + 1 = 8, \ldots, p_l + 1$$



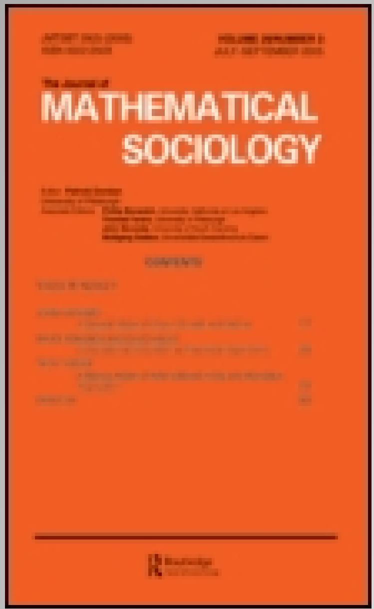
So by considering the global partition $\tau = \bigcup_{k=1}^{l} \tau_k$

The period of the network is $T \geq \prod_{k=1}^{l} p_k = e^{\Omega\left(\sqrt{|V(G)|\log|V(G)|}\right)}$

Same arguments can be done for the transient time.

The SAKODA's model of social Interactions

Complexity2017 (Medina, E.G., R. Zarama, S. Rica)



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The checkerboard model of social interaction

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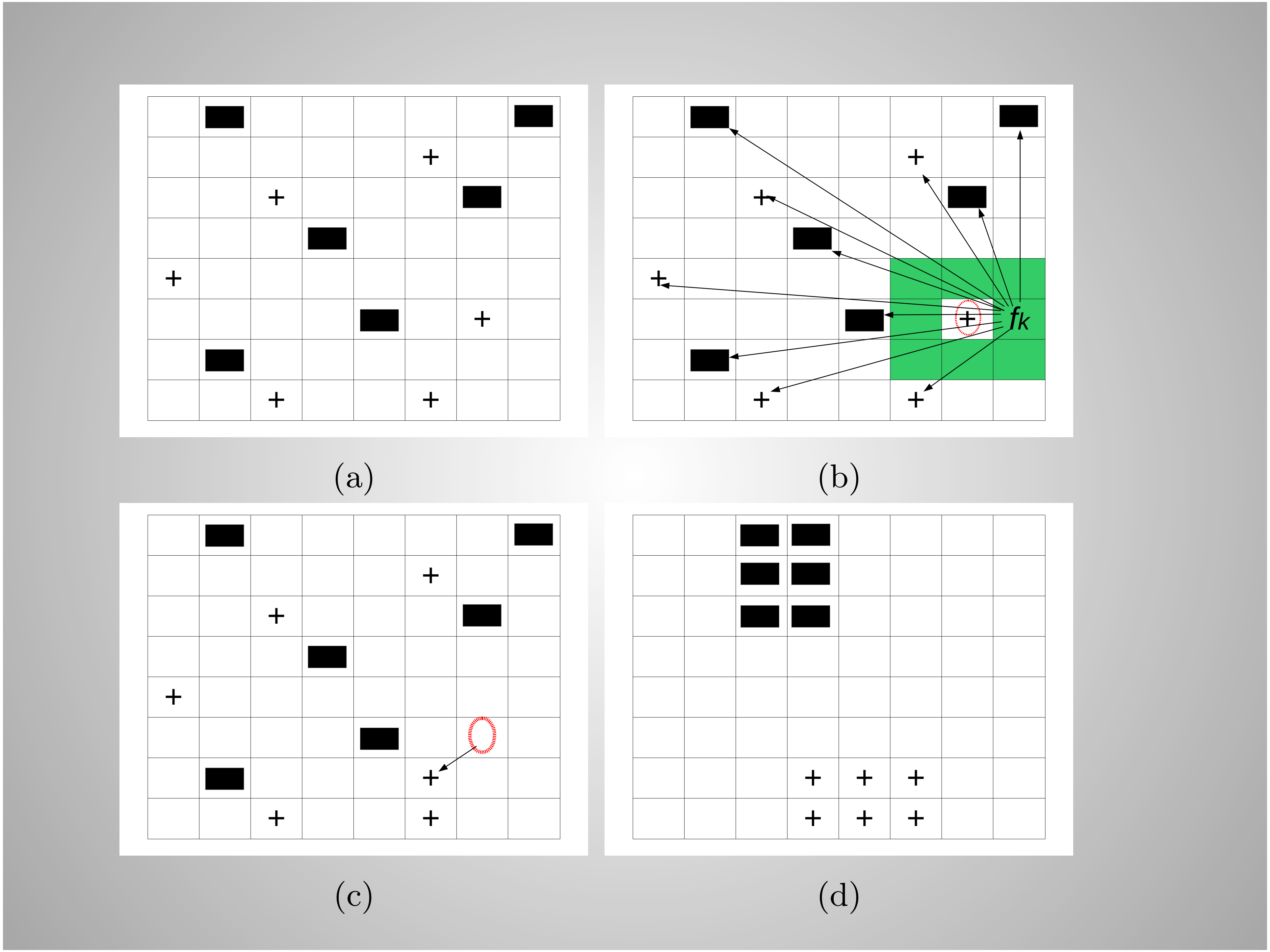
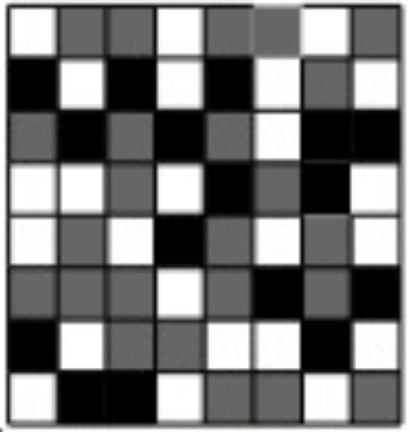
,Sergio Rica (Físico, Universidad Adolfo Ibáñez, Santiago, Chile
Roberto Zarama , Ingeniería Industrial, Universidad de los Andes, Bogotá
EG., Matemático, Universidad Adolfo Ibáñez, Santiago, Chile.


f_k
(a)
(b)
(c)
(d)



1. The simulation starts with a random configuration with a fraction ϕ_V of empty nodes (gray nodes) and fraction ϕ_{+1}, ϕ_{-1} of occupied nodes in +1 (white) and -1 (black) states respectively

Nodes and states: *A node k occupied by an individual at state* $x_k = +1, -1$

Fract. vaccancy and occupancy: Vaccancy ϕ_V, occupancy $1-\phi_V$
With $1-\phi_V = \phi_{+1} + \phi_{-1}$ *and* $\phi_{\pm 1}$ *the fraction of states* ± 1 *respectively* $(\phi_{+1} = \phi_{-1})$

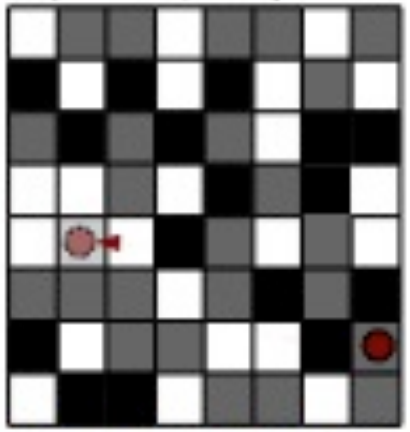
The *S*-matrix

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \; ; \; s_{\alpha\beta} = \begin{cases} -1 & \text{repulsion between} \quad x_k, x_j \\ 0 & \text{neutrality between} \quad x_k, x_j \\ +1 & \text{atraction between} \quad x_k, x_j \end{cases}$$

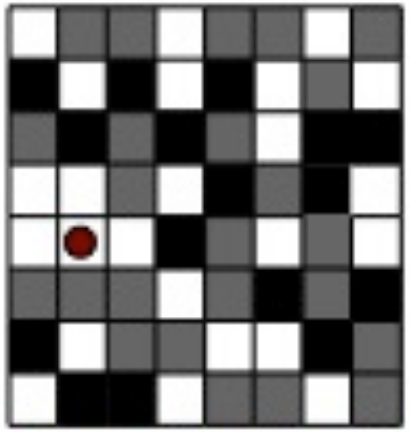
The potential *fk*

$$f_k = \sum_j^N \delta(x_k, x_j)\, v(r_{kj}); \quad \delta(x_k, x_j) = \begin{cases} s_{\alpha\beta} & \text{for} \quad x_k, x_j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

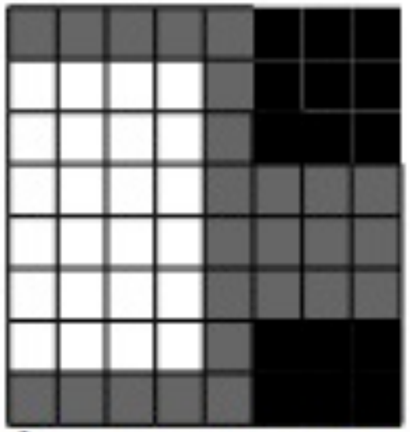
$v(r_{kj})$ *A function that depends on the distance between node k and j*



2. An individual moves different empty node considering a long move (all network) or short movement (neighborhood) move and evaluate the potential f_k considering long influence or short influence



3. The individual move to the position in its neighborhood that improves the value of the potential f_k



4. The steps two and three are applied to all individuals recursively many times. The system converges to an attractor after various iterations of these steps.

Sakoda's attitude matrices

$$S = \begin{array}{cc} & \begin{array}{cc} \circ(+1) & \bullet(-1) \end{array} \\ \begin{array}{c} \circ(+1) \\ \bullet(-1) \end{array} & \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} \end{array}$$

-1, 0, 1 entries

There are 81 matrices. By diagonal symmetries (changing blacks and whites) only 45 different.

$$= \begin{pmatrix} \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}.$$

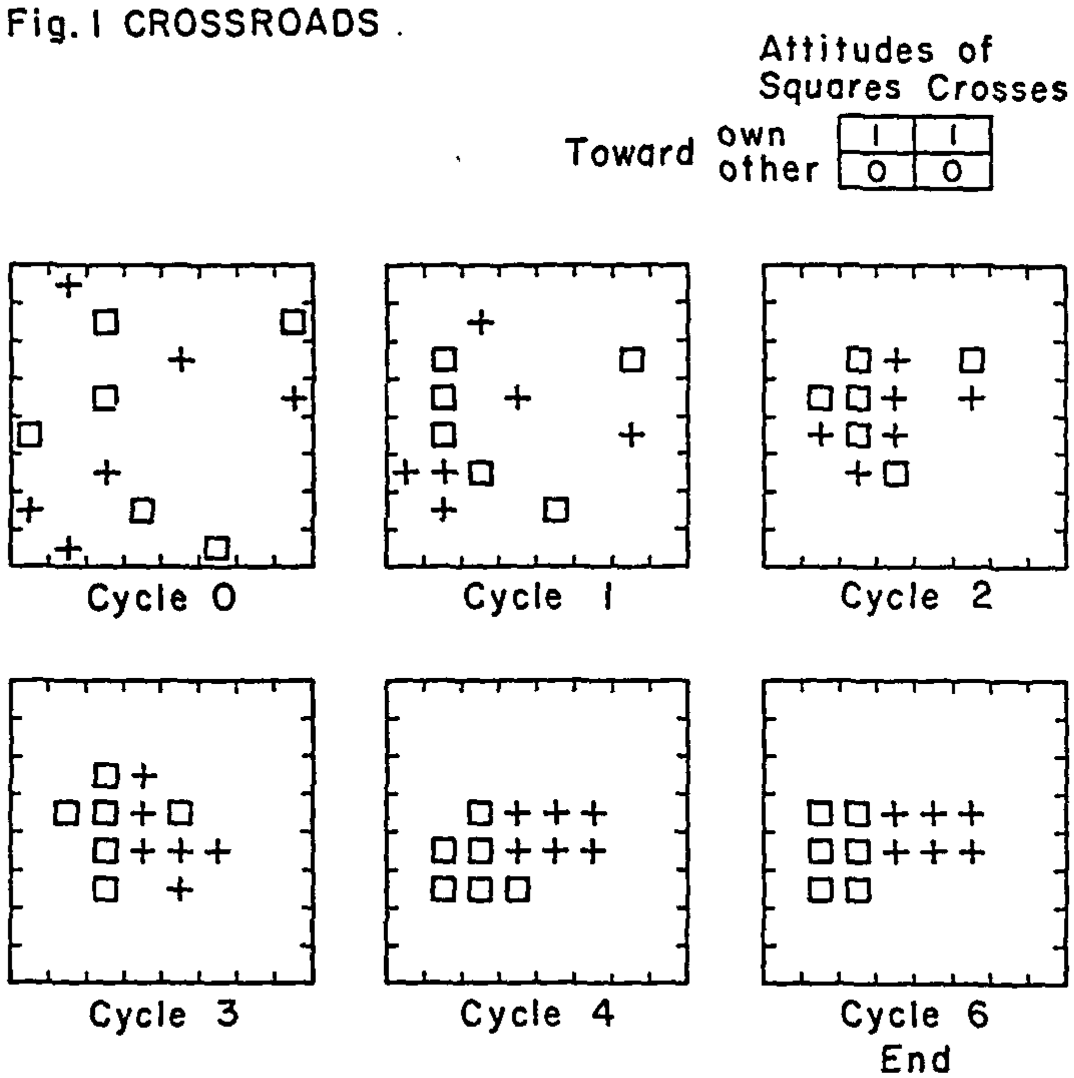


FIGURE 1 *Crossroads* (Squares: 1 to Own, 0 to Other; Crosses: 1 to Own, 0 to Other).

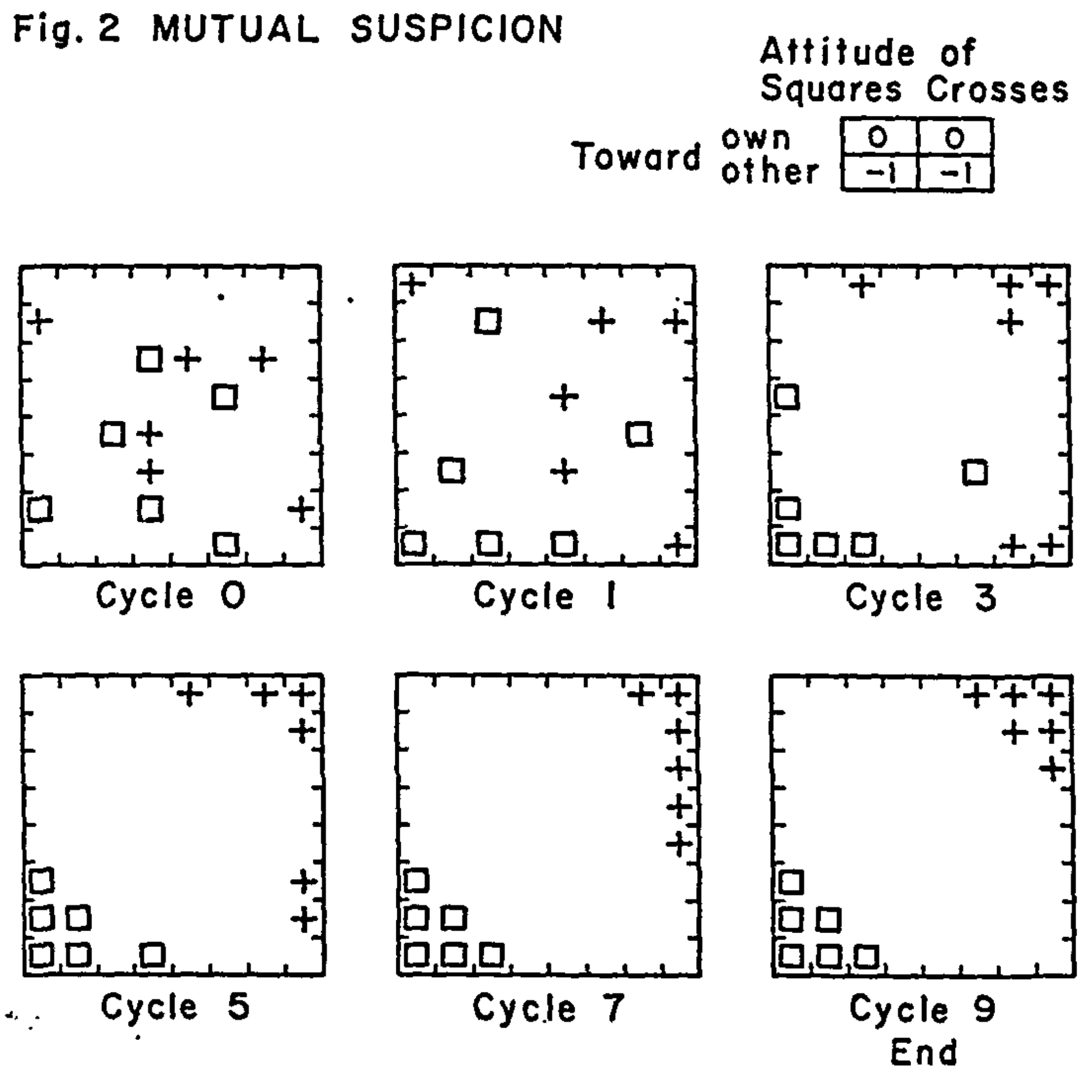


FIGURE 2 *Mutual Suspicion* (Squares: 0 to Own, −1 to Others; Crosses: 0 to Own, −1 to Others).

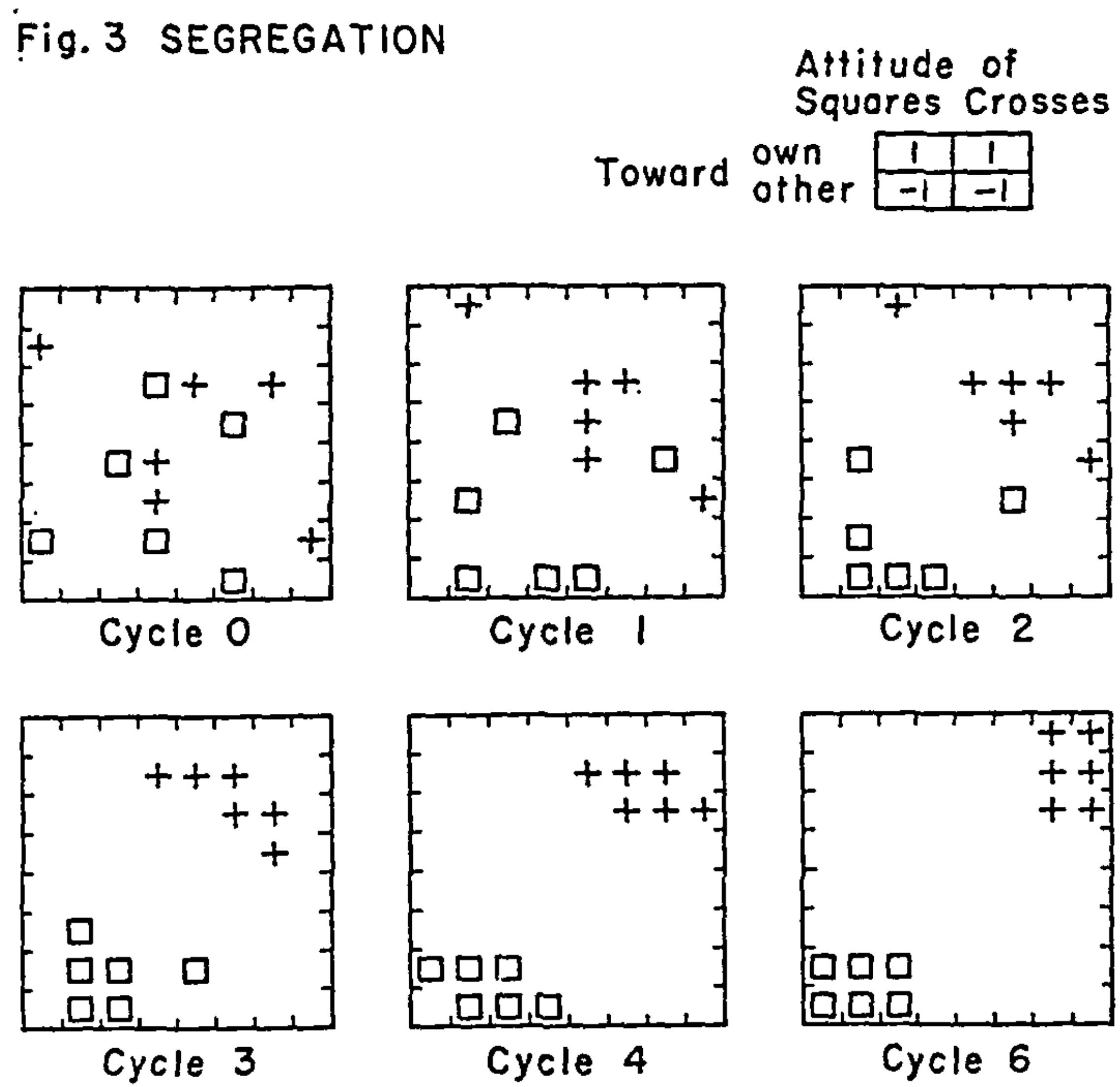


FIGURE 3 *Segregated Groups* (Squares: 1 to Own, −1 to Others, Crosses: 1 to Own, −1 to Others).

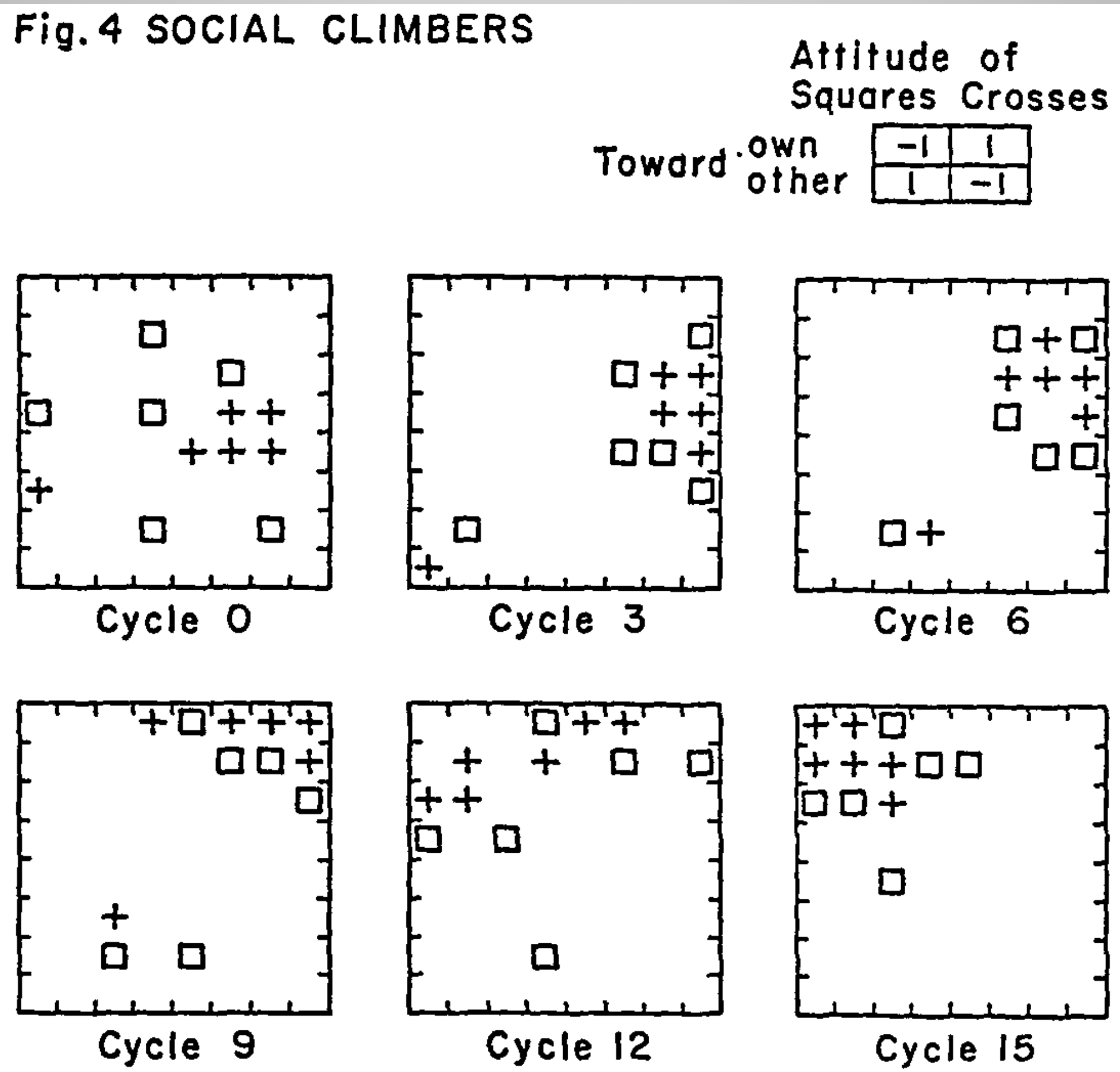


FIGURE 4 *Social Climber* (Squares: −1 to Own, 1 to Other; Crosses: 1 to Own, −1 to Other).

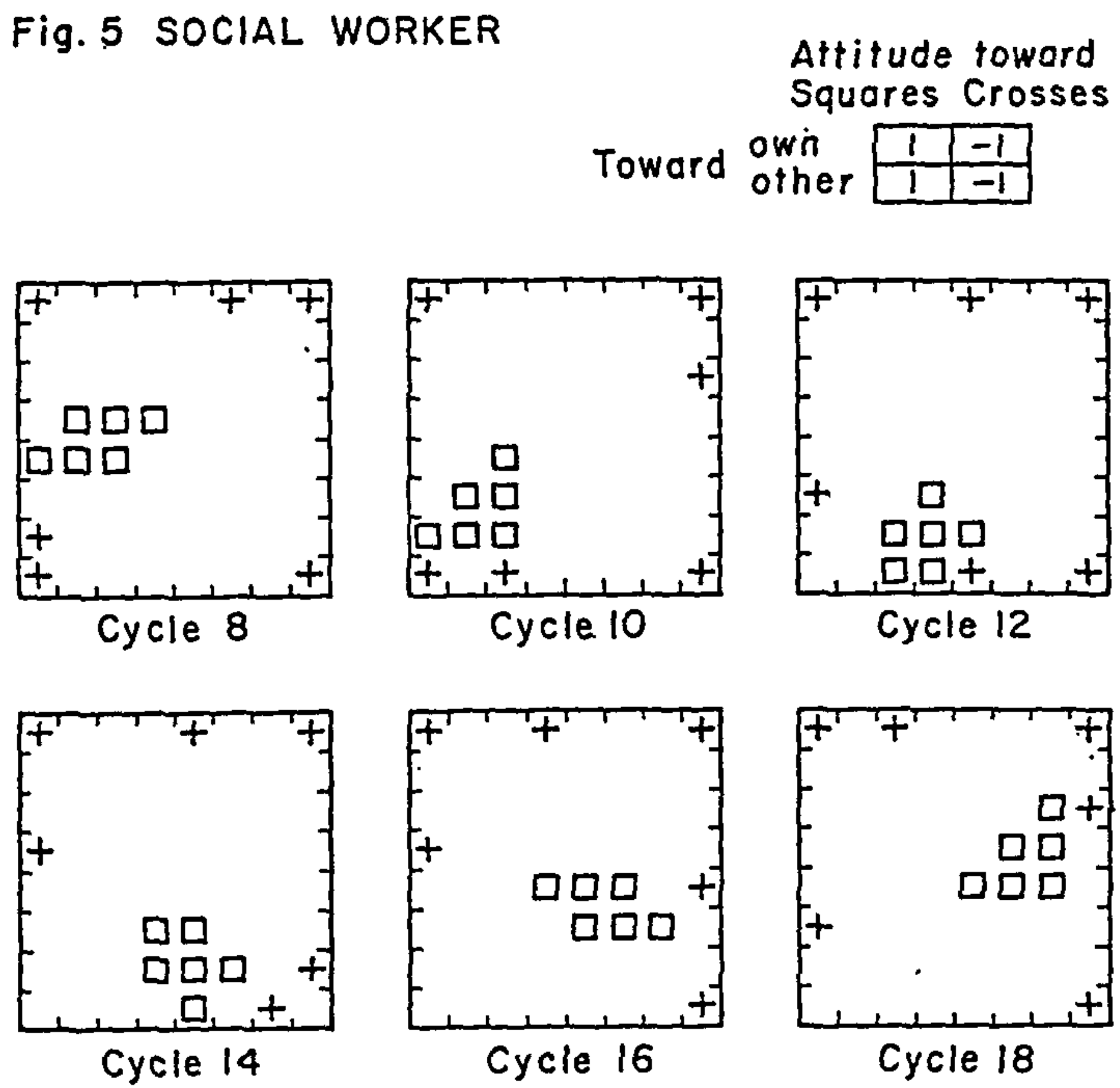


FIGURE 5 *Social Worker* (Squares: 1 to Own, 1 to Other; Crosses: −1 to Own, −1 to Other).

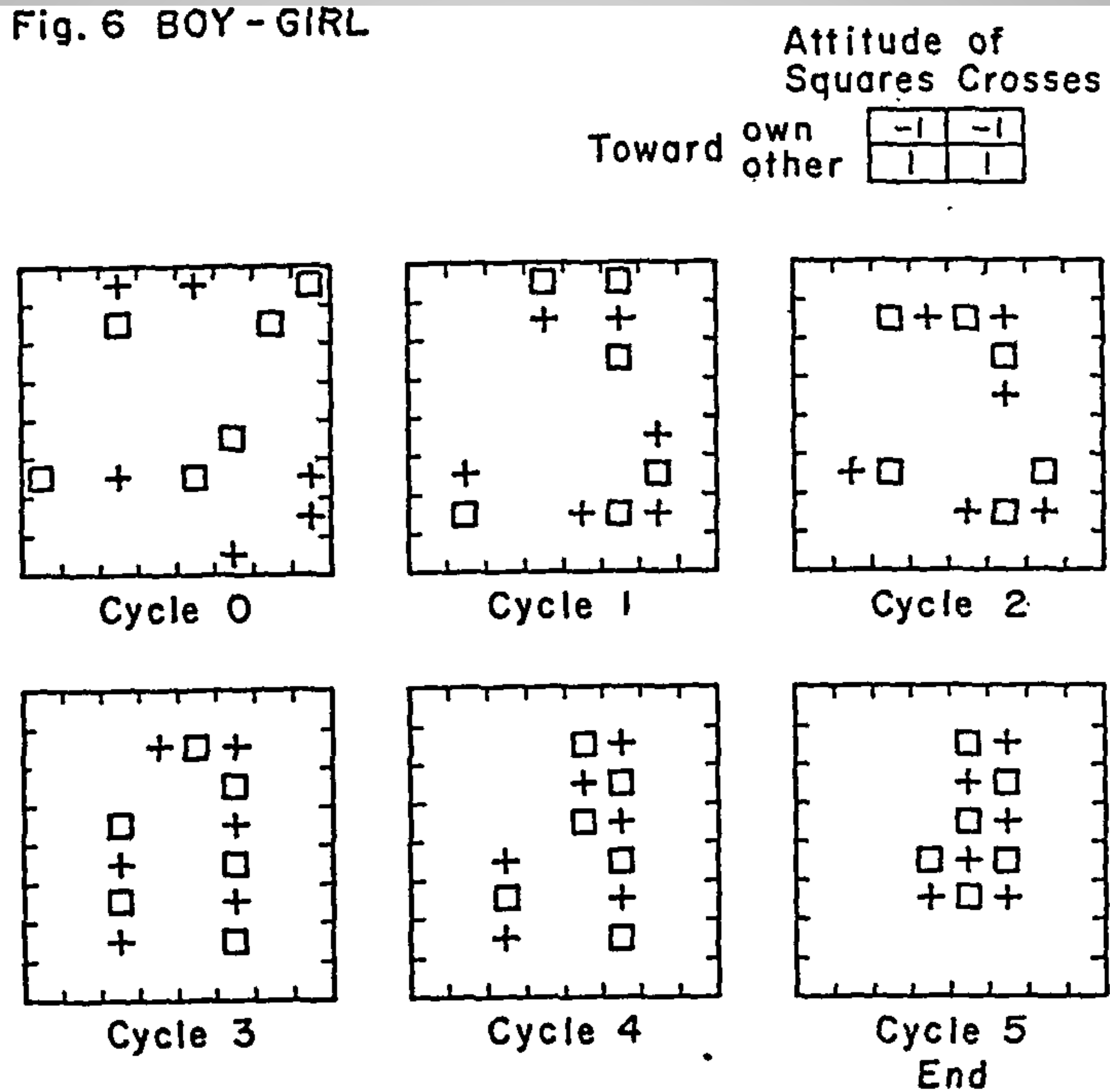


FIGURE 6 *Boy-Girl Situation* (Squares: −1 to Own, 1 to Other; Crosses: −1 to Own, 1 to Other).

The local field:

$$f_k = \sum_{j=1}^{N} \delta(x_k, x_j) v(r_{kj})$$

$$\delta(x_k, x_j) \quad = \quad \begin{cases} s_{\alpha\beta} & \text{for } x_k, x_j \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

The Energy:

$$E[x] = -\frac{1}{2} \sum_k f_k = -\frac{1}{2} \sum_{k,j} \delta(x_k, x_j) v(r_{kj})$$

$$v(r_{kj}) = \frac{1}{r_{kj}^2}$$

E is decreasing when S is symmetric and (in general cases of v functions)

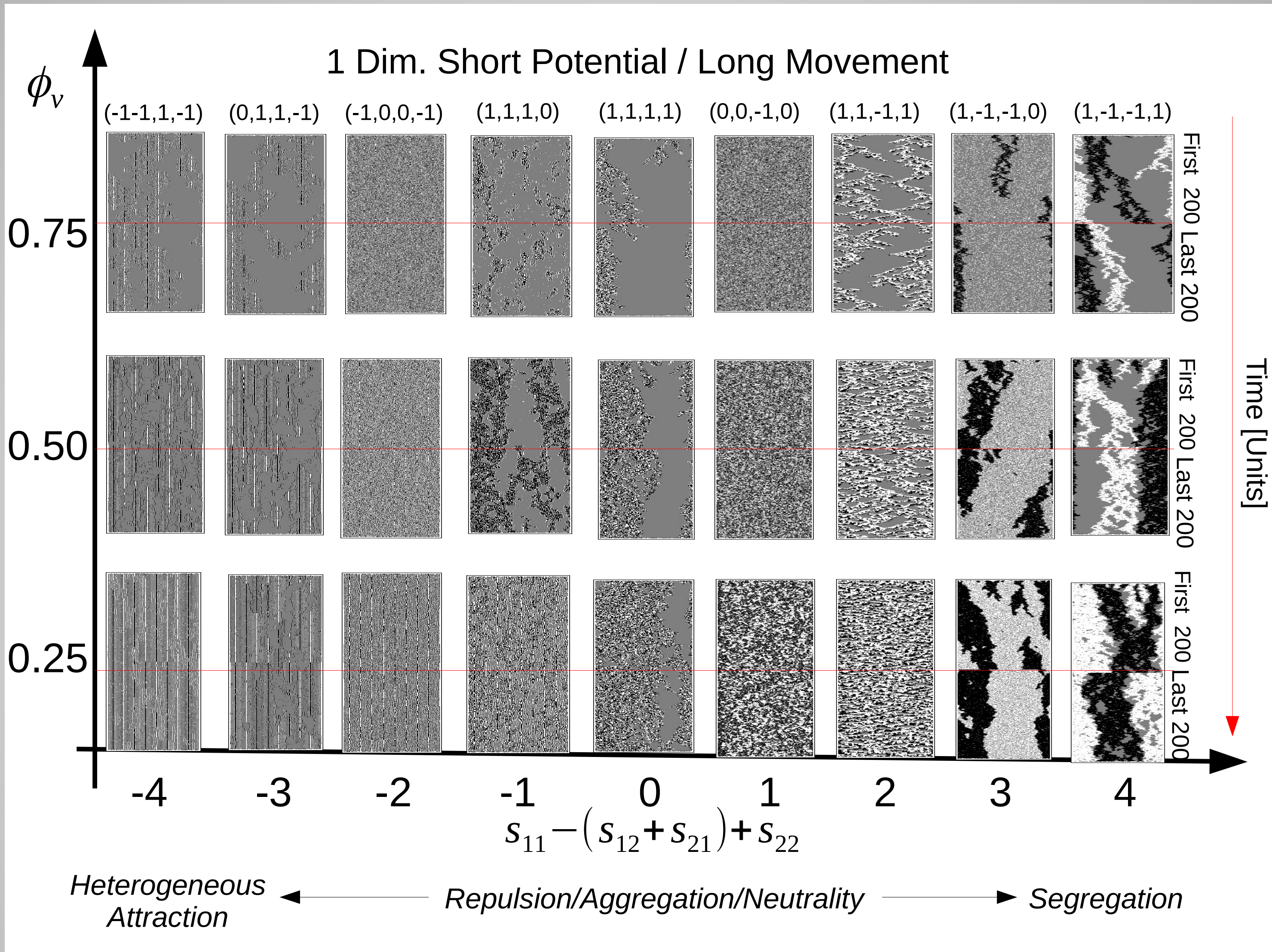


Figure 5: Phase diagram for different values of ϕ_V and χ considering Short influence potential/Long range movement in a one dimensional lattice. This attractors are the results of simulations considering 128 one dimensional lattice. The time goes from top to bottom; the red line divides between the first 200 evolution steps and the final 200 evolution steps. We consider preiodic boudary conditions.

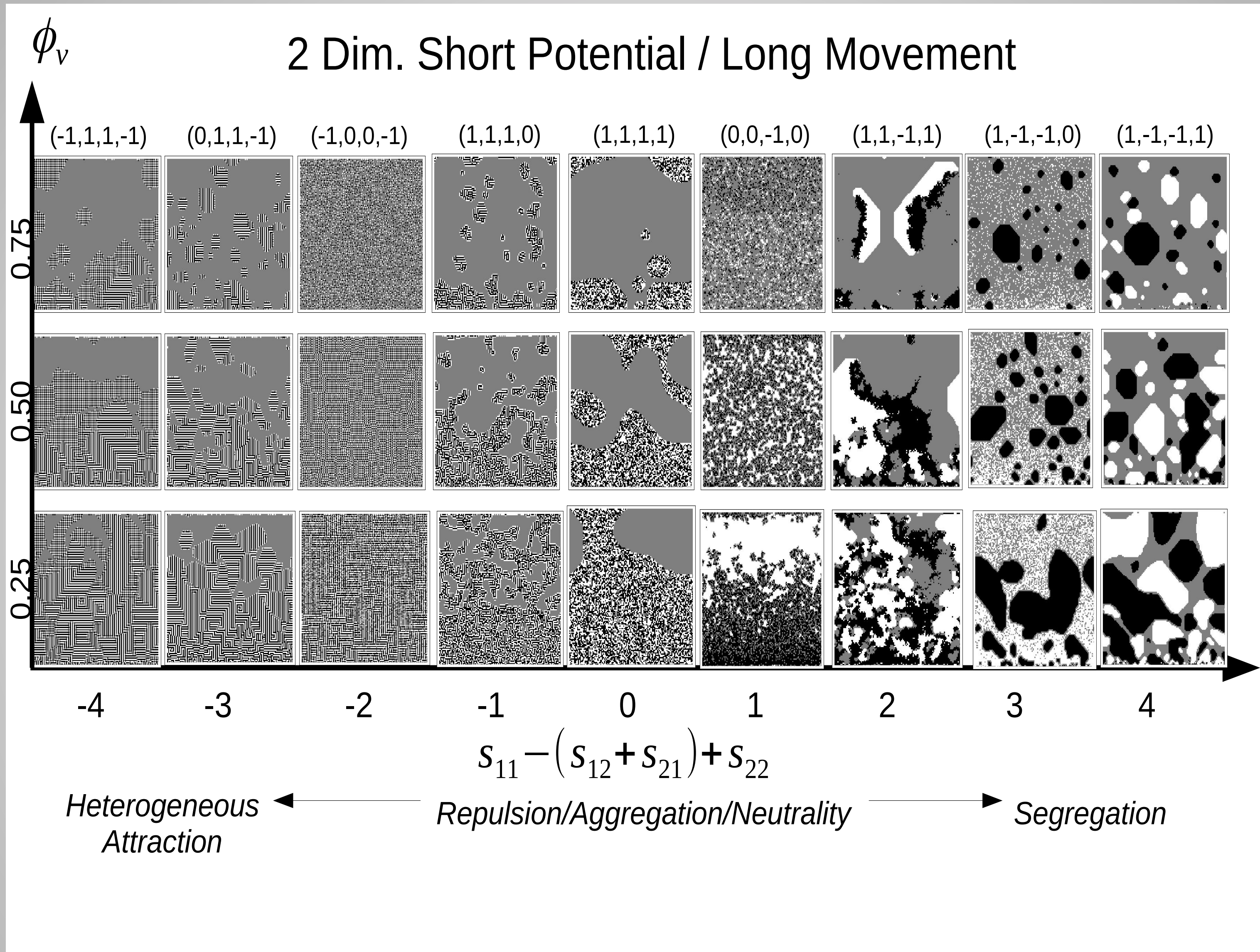
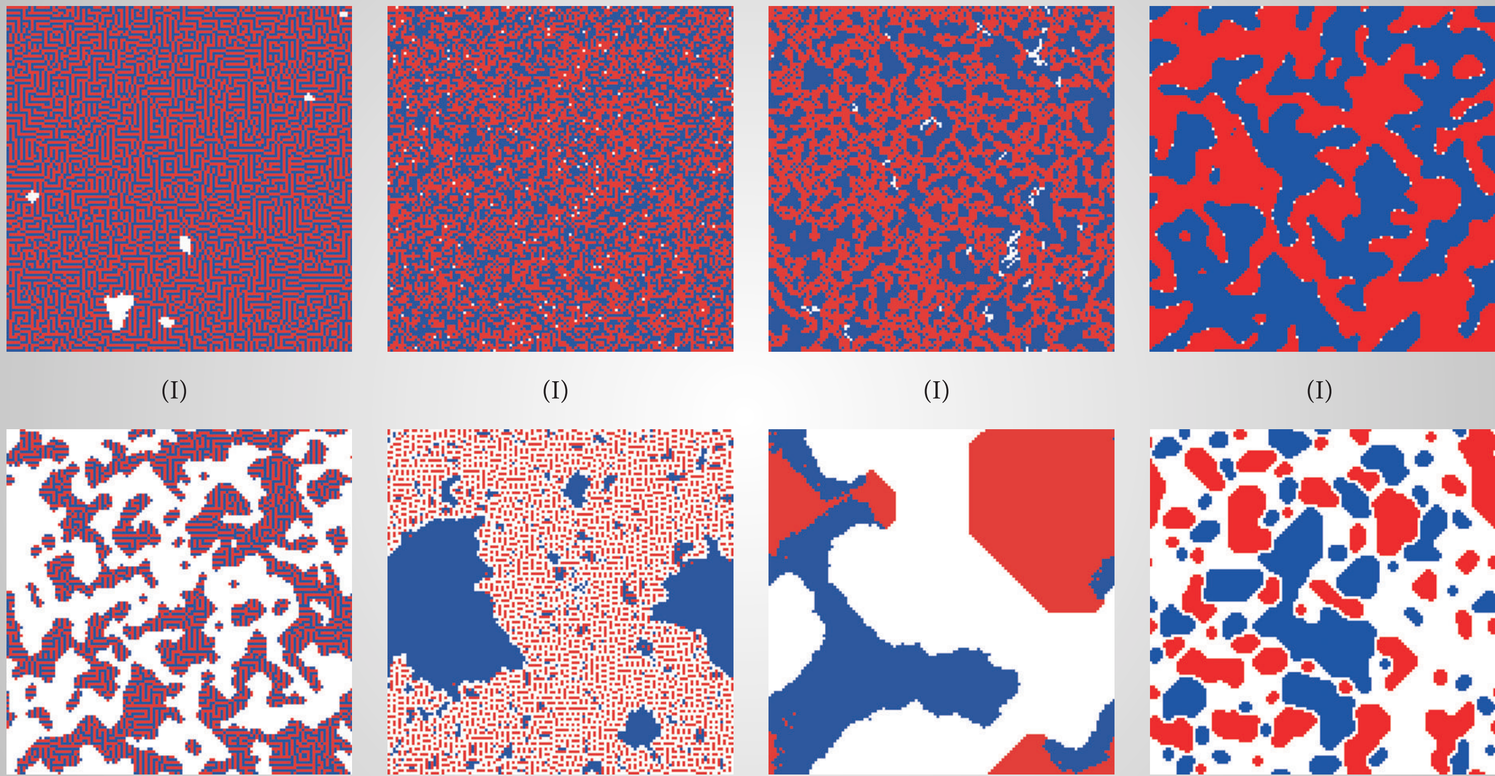


Figure 13: Different interaction types (attractors) in terms of the entries of the $S-$matrix in 128×128 two dimensional lattice considering Moore neighborhood and boundary periodic conditions.



(I)
(I)
(I)
(I)

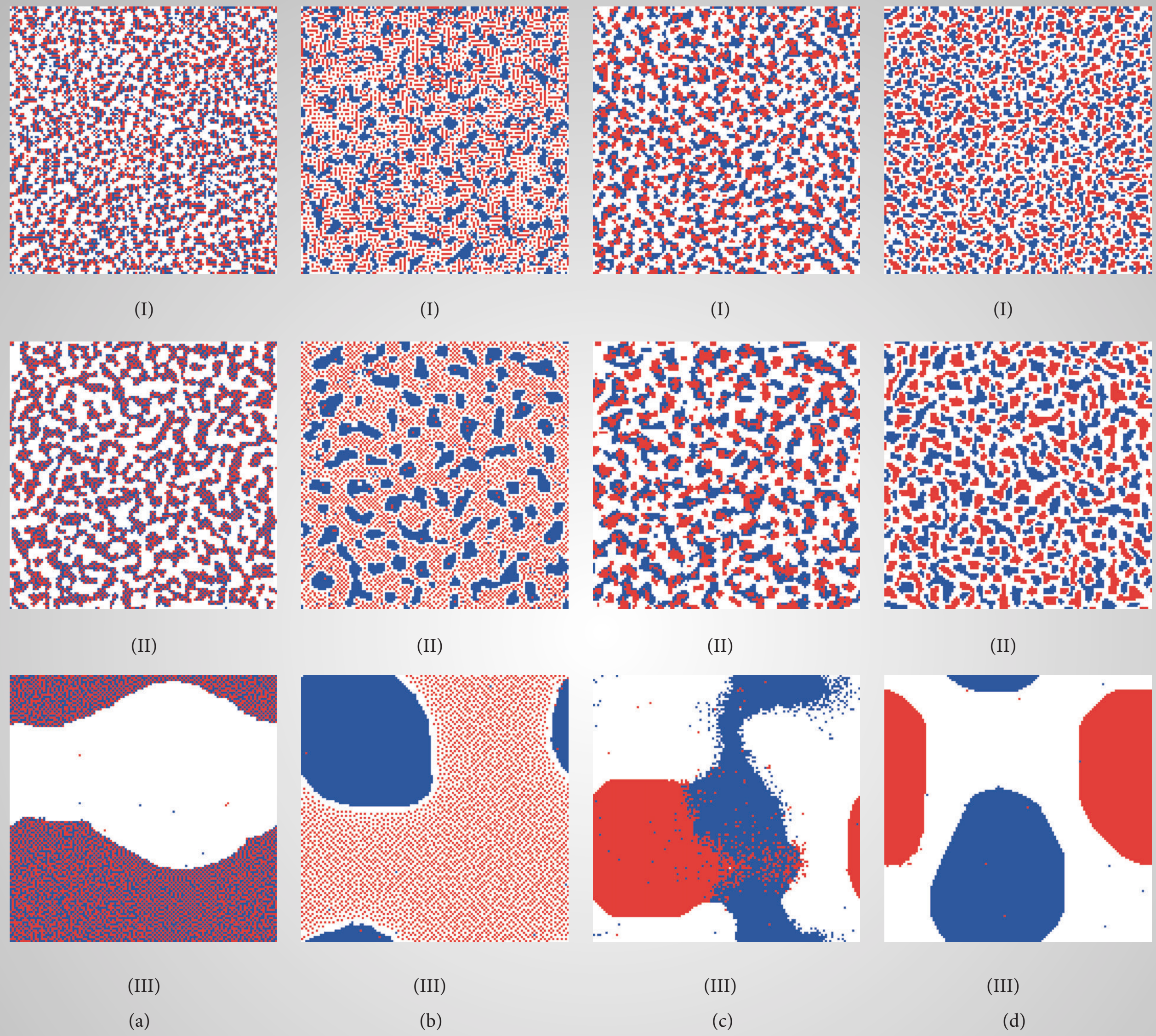


FIGURE 7: Snapshots for different interaction and movement (rest) rules for the same cases as in Figure 1. (a) $S = (0, 1, 1, 0)$, (b) $S = (1, 1, -1, -1)$, (c) $S = (1, 1, -1, 1)$, and (d) $S = (1, -1, -1, 1)$. The first line (I) corresponds to the case of short range interaction and short range movement. The second row (II) corresponds to long range interaction/short range movement, and, finally the third line (III) corresponds to the case of long range interaction and long range movement. Simulations were run in a $N = 128 \times 128$ lattice with $\phi_V = 0.5$ and for 2×10^5 steps for (I) and (II) and 5×10^4 steps for (III).

The Schelling Segregation model

(Nicolás Goles-Domic, Sergio Rica, E.G.

PHYSICAL REVIEW E **83, 056111 (2011)**
And work in progress.

chapter 4:

Sorting and Mixing

DYNAMIC MODELS OF SEGREGATION†

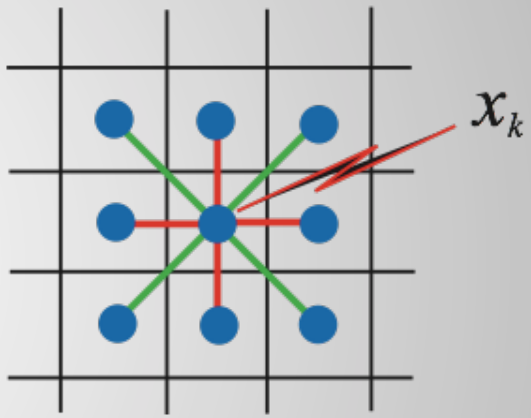
THOMAS C. SCHELLING
Harvard University

Journal of Mathematical Sociology
1971, Vol. 1, pp 143–186

The Model of Segregation by Shelling

Thomas C. Schelling (1969 - 1972)

- Lattice one or two dimensional with periodic conditions
- State $x_k = \pm 1$

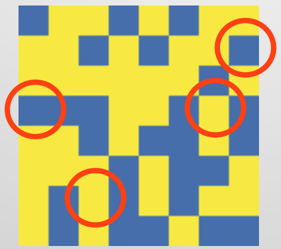


- Neighborhood Moore (green and red arrows) and von Neumann (red arrows)
- Tolerance threshold $\theta \in \{1, \dots |V|\}$

Happiness threshold

An individual is unhappy if there are more than θ individuals on the other state in its neighborhood

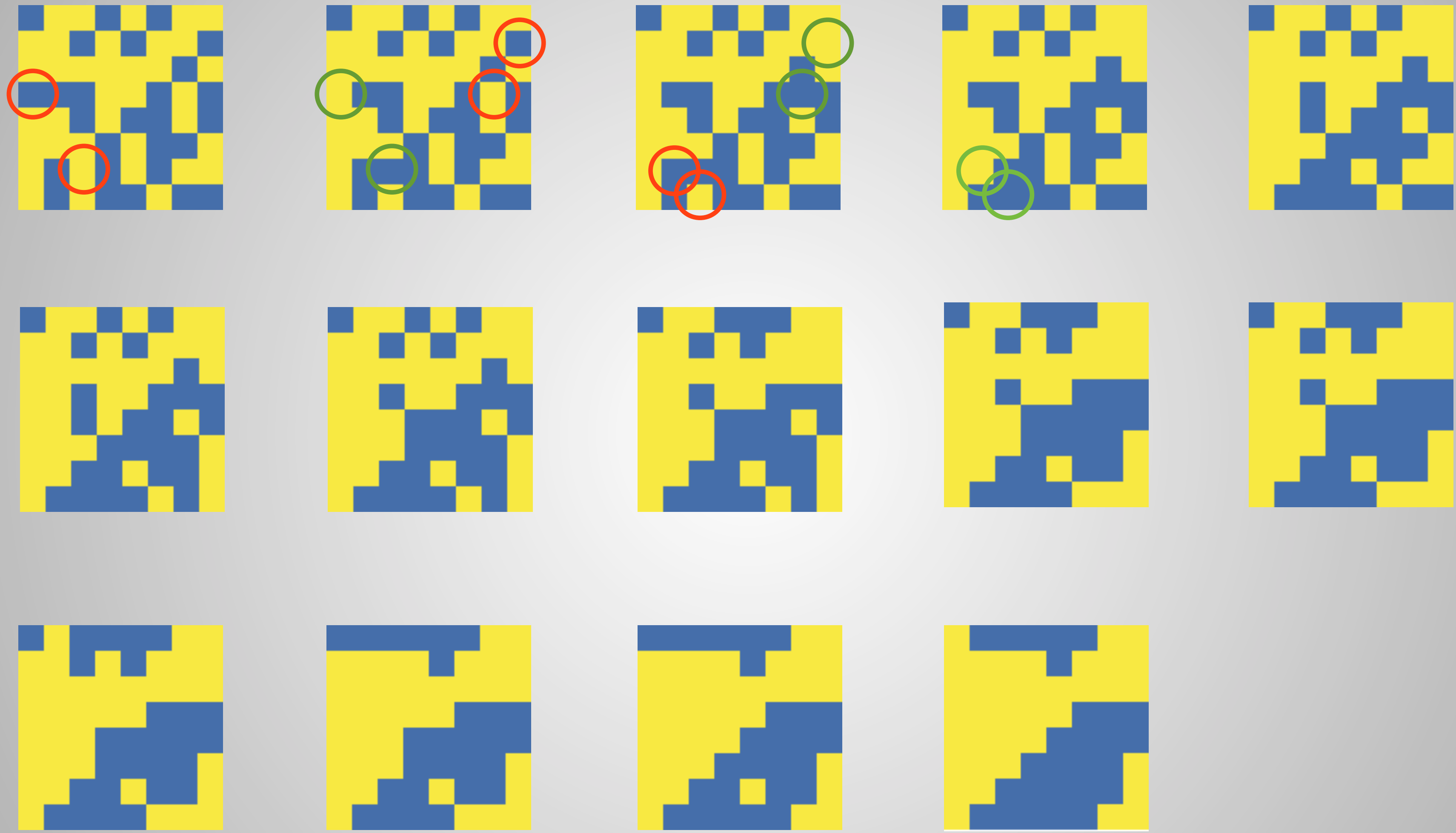
eg. For the Moore's neighborhood and $\theta = 5$ then :



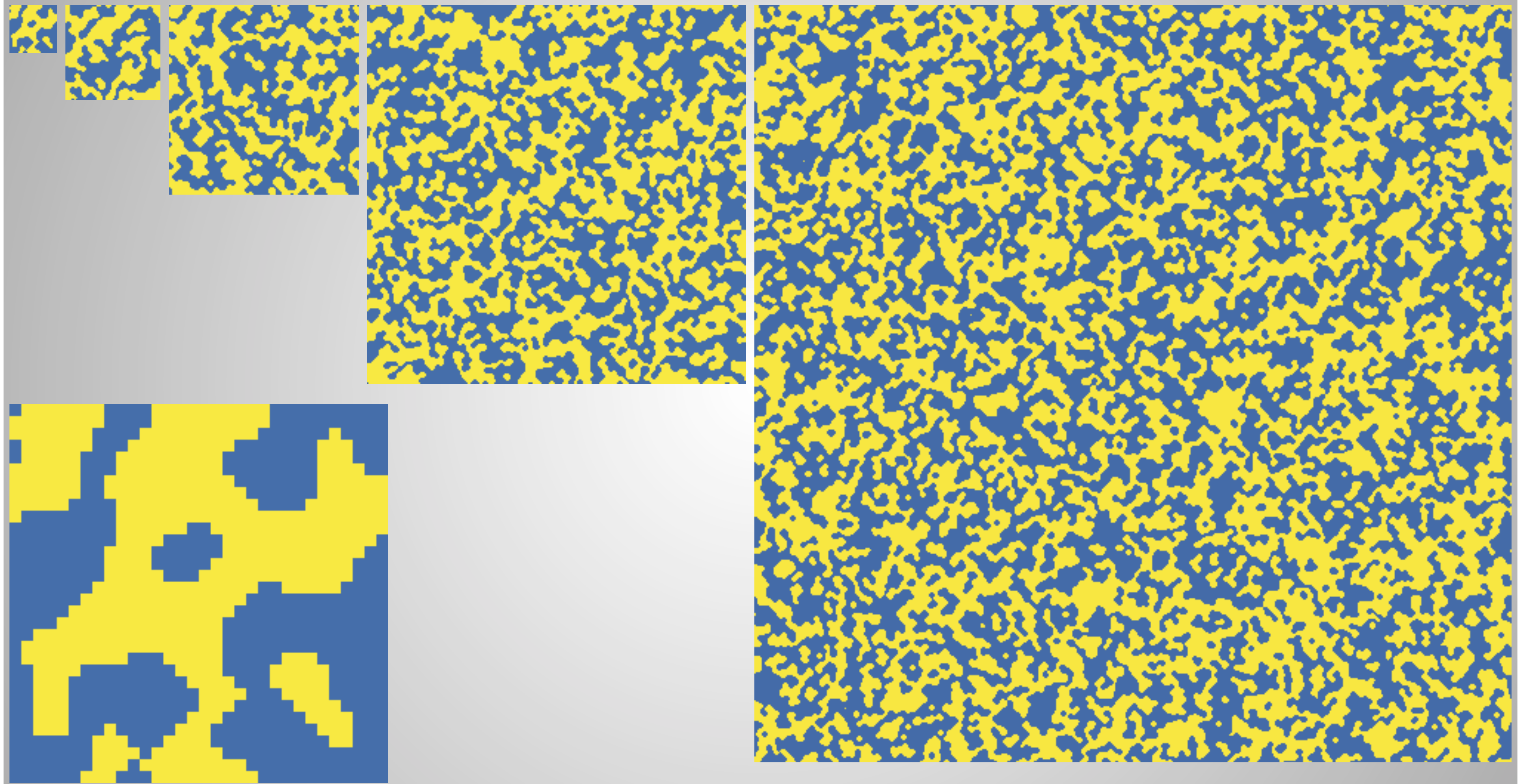
The update rule

At each step, one lists the unhappy individuals of both species, and then randomly (for instance) one exchanges two individuals of opposite value.

$\theta = 5$

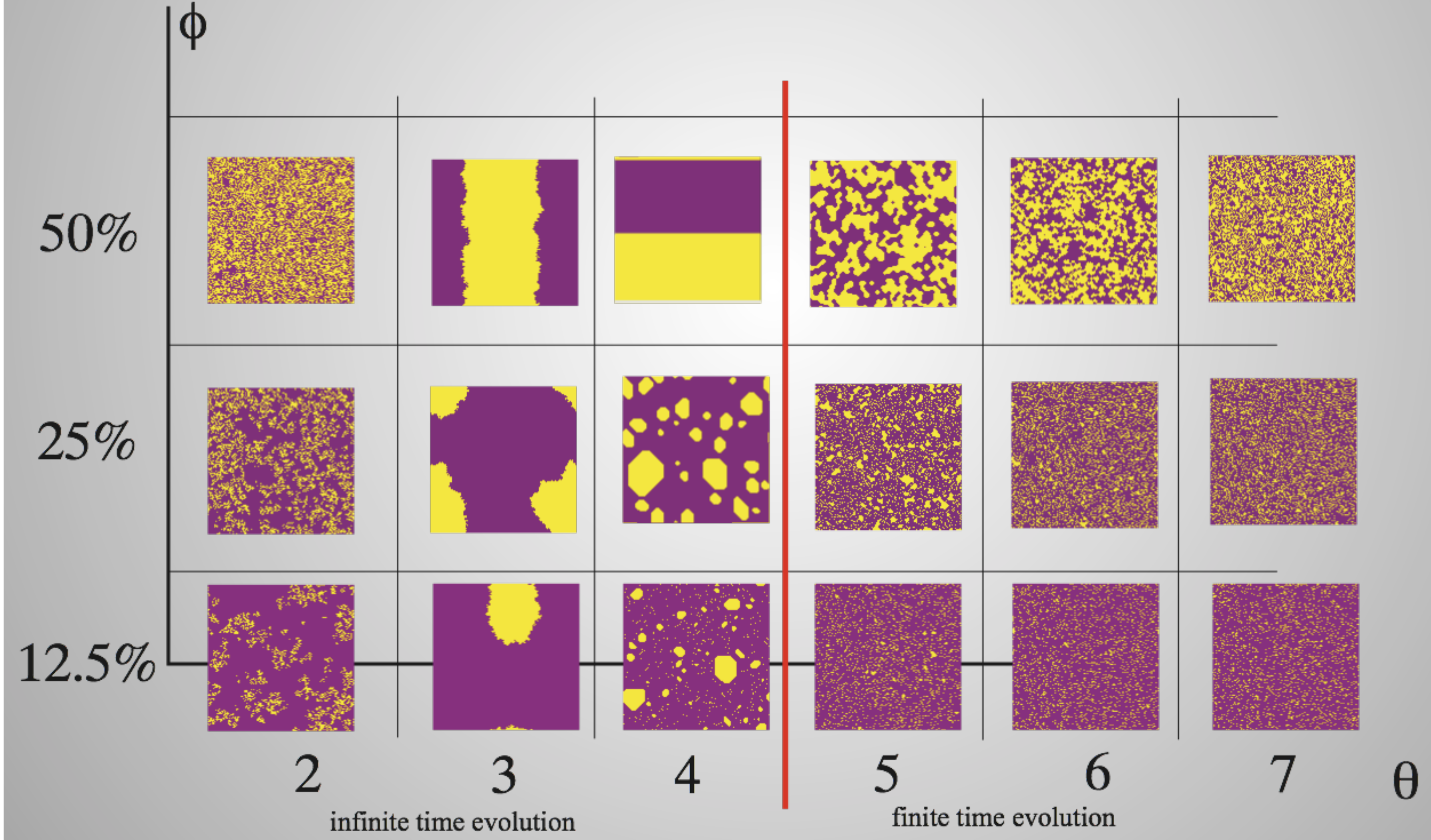


Length scale for $\theta = 5$



32 x 32 64 x 64 128 x 128 256 x 256 512 x 512

Phase diagram
for Moore's neighborhood



Quantitative behavior

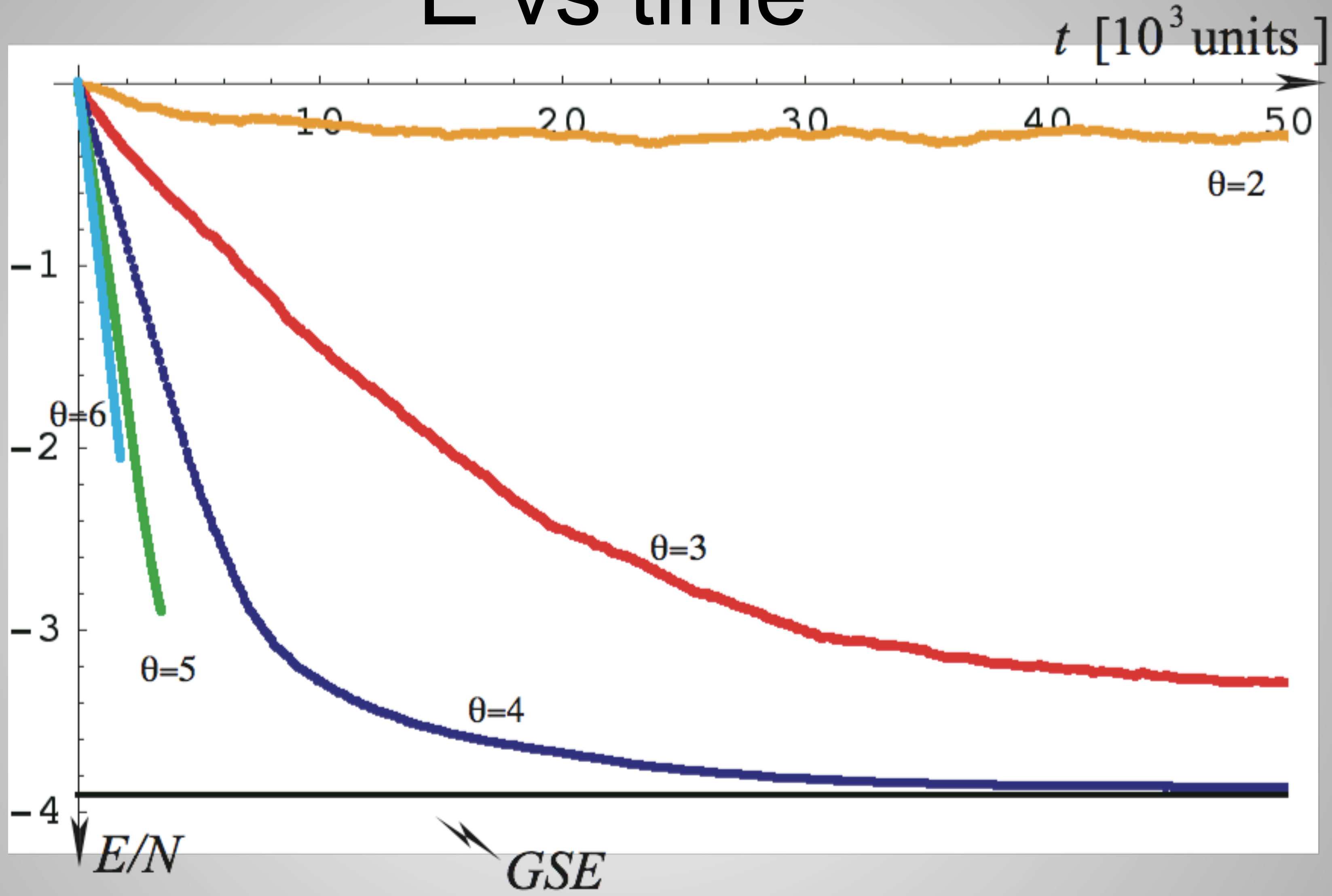
$\theta \geq 5$:the energy decreases

$$E[\{x\}] = -\frac{1}{2}\sum_{k=1}^{N} x_k \sum_{i \in V_k} x_i$$

In general, if V is the neighborhood, the energy decreases
If and only if

$$\theta > \frac{|V|}{2}$$

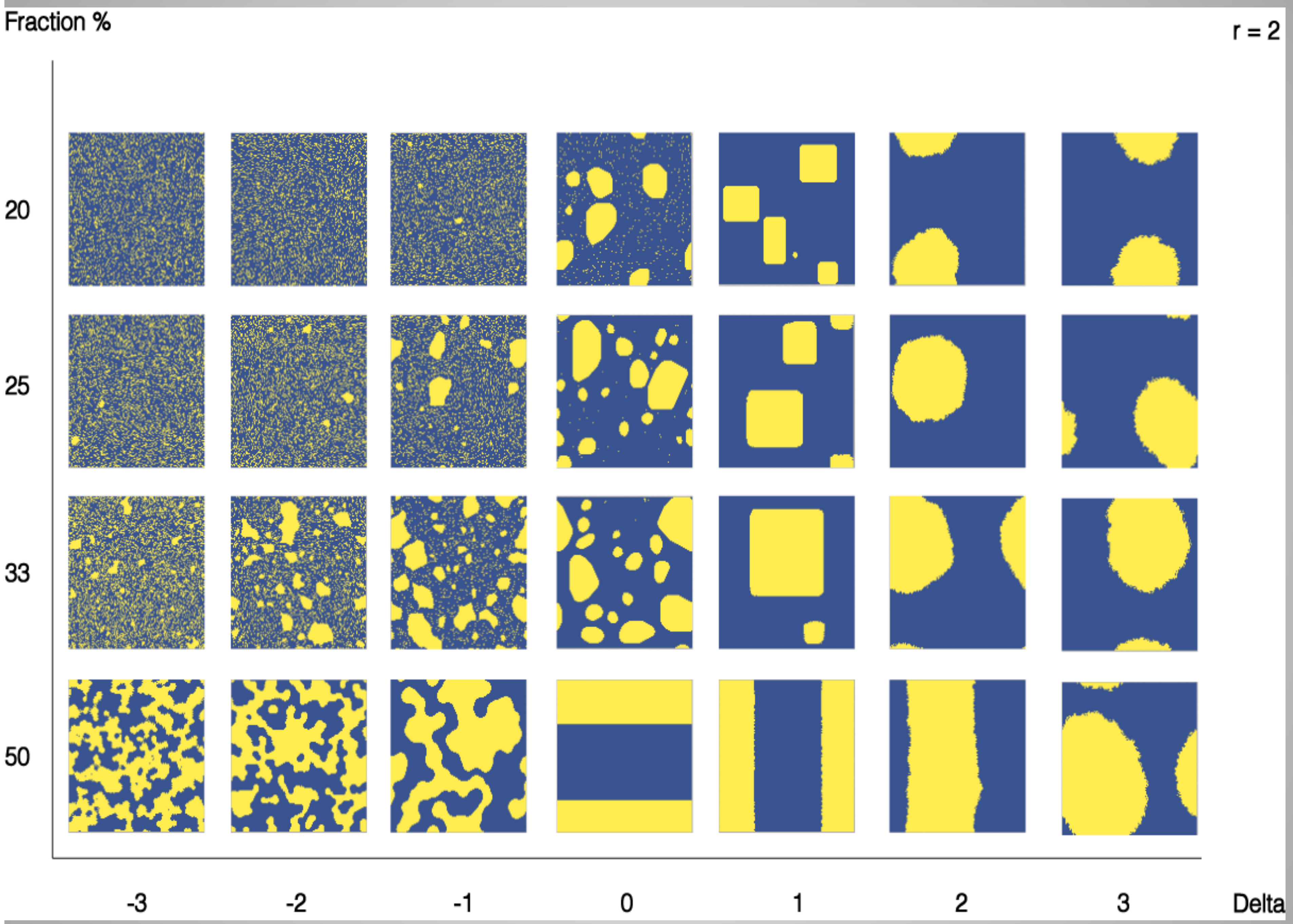
E vs time



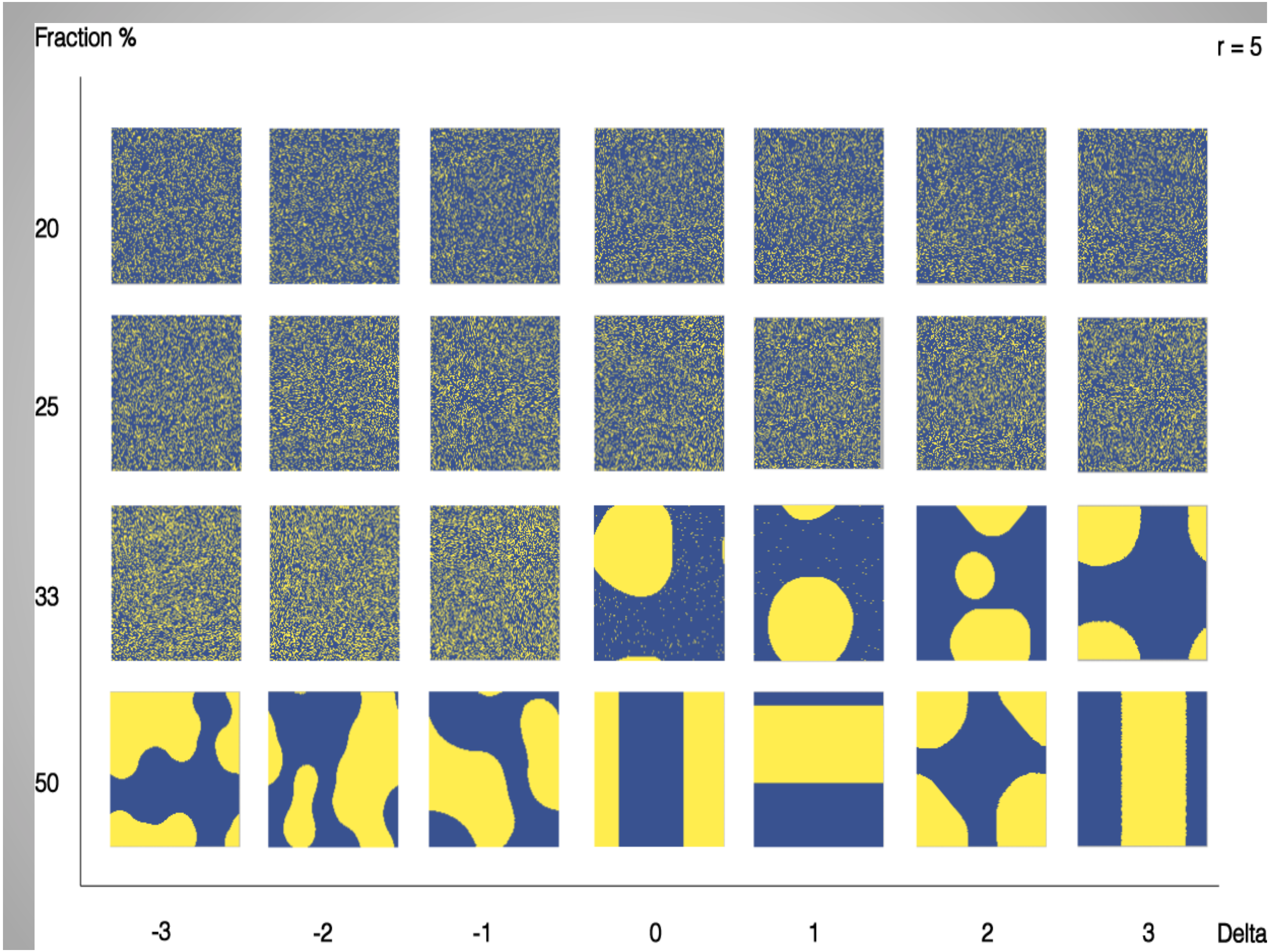
Geometrical interpretation

It is easy to see by a transformation of the energy that
Minimize it, is equivalent to minimize the perimeter of
The clusters …… so the dynamics try to do that !!

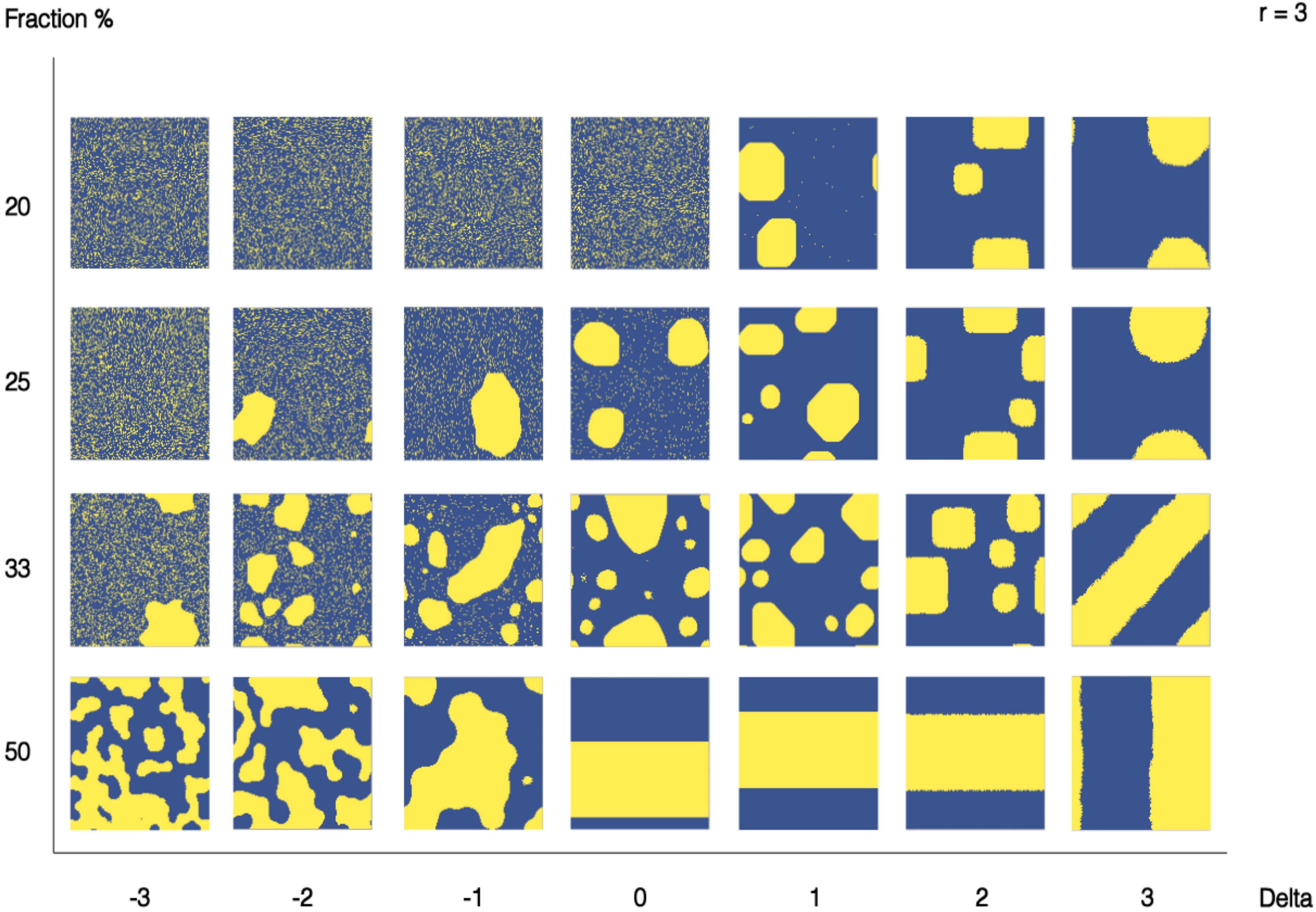
Others phase diagrams with circle-neighborhoods
with different radios (Nicolas Goles-Domic Simulations):



Fraction %
r = 2
20
25
33
50
-3
-2
-1
0
1
2
3
Delta



Fraction %
r = 5
20
25
33
50
-3
-2
-1
0
1
2
3
Delta



Fraction %
r = 3
20
25
33
50
-3
-2
-1
0
1
2
3
Delta

Regulatory Boolean networks

- Stuart. Kauffman,Metabolic stability and epigenesis in randomly connected nets, J. Of Theor. Biol, 22, 437-67, 1969.
- René Thomas, "Regulatory networks seen as asynchronous automata : a logical description.", *J. Theor. Biol.* **153** ,(1991) 1-23
- François Robert, Discrete Iterations, Springer Verlag, 1986).

The dynamics may change a lot for different updates.

Cycles may appear or dissapear.

Given a regulation network what is the “right” way to update it?.

Conditions such that there exists only fixed points.

Condition for the robustness of the network (roughly speaking, same behavior for different updates).

Since there are an exponential number of updates
To study all them implies a huge amount of computation
So it is important to determine tools in order to diminish
the computing time .....

We will present here *dynamical equivalence classes*
and the notion of *alliances*

Previous notions where defined to study yeast's regulation networks

Below we will always consider threshold interactions

$$x'_i = 0 \Leftrightarrow \sum_{j=1}^{n} w_{ij} x_j - \theta_i \leq 0$$

1 otherwise

We consider threshold interactions

$$x'_i = 0 \Leftrightarrow \sum_{j=1}^{n} w_{ij} x_j - \theta_i \leq 0$$

1 otherwise

Equivalent classes: given a regulatory network (F,G) and the updates modes s_1 and s_2 we obtain two dynamics:

$$(F, s_1) \quad \text{and} \quad (F, s_2)$$

They are equivalent if and only if its phase space is the same.

There are "good" algorithms to determine equivalent classes for reasonable size regulatory networks (say n no more than 15 and no to much interconnections.

Aracena, J., E.G, E., Moreira, A., & Salinas, L. (2009).
On the robustness of update schedules in Boolean networks. *Biosystems*, *97*, 1–8.

Montalva, M. (2011). *Feedback set problems and dynamical behavior in regulatory networks*. PhD, Universidad de Concepción, Concepción, Chile.

Alliances: Given a regulatory network (F,G)

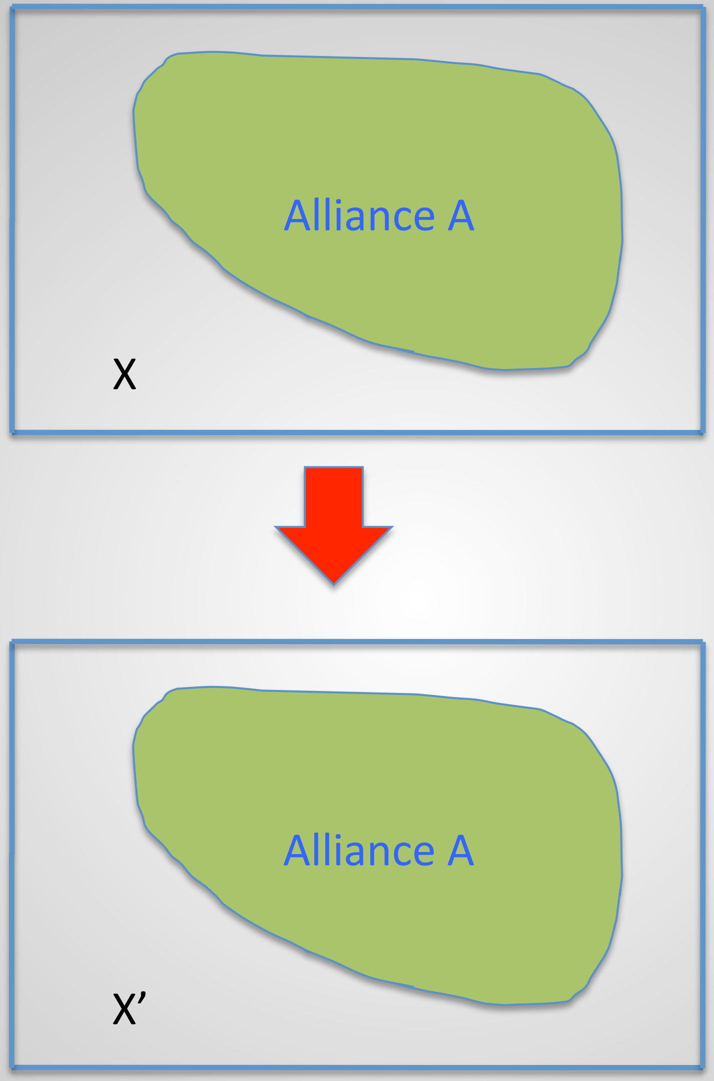
We say that a subconfiguration $(a_{i_1},\ldots,a_{i_p})\in\{0,1\}^p$ Is an *alliance* if and only if

$$a_{i_k} = H\left(\sum_{j=1}^{p} w_{i_k i_j} a_{i_j} + \sum_{j\notin\{i_1,\ldots,i_p\}} w_{i_k j} x_j - \theta_{i_k}\right) \qquad k\in\{1,\ldots,p\}$$

$$x\in\{0,1\}^{n-p}$$

An *alliance* are a kind of “local” fixed point but stronger….

It is invariant under updates changes



After update

EXAMPLE1

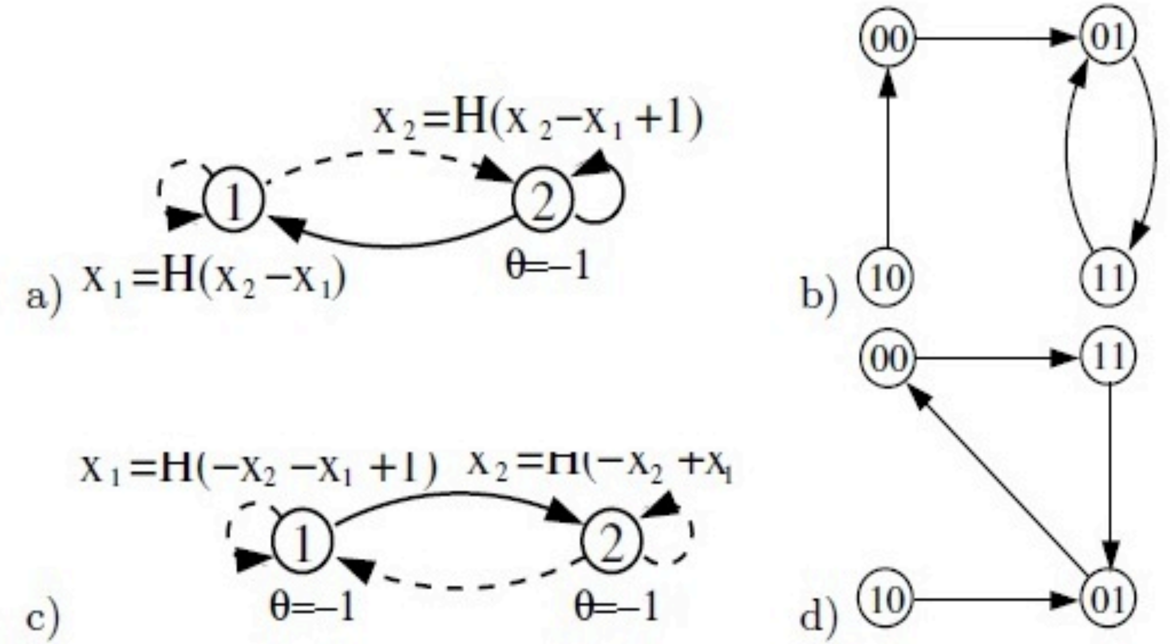


Fig. 1 In figures a) and c), the dotted lines represent negative interactions (weight$= -1$) while the others are positive interactions (weight$= 1$). a) A BN of 2 nodes with threshold Boolean functions defined as x_1, x_2, and thresholds $\theta = 0$ and $\theta = -1$ respectively. b) The dynamical behavior of a) for the parallel update schedule. Note that $x_2 = 1$ is an alliance although the dynamic has no fixed point. c) A BN as in a) but with both thresholds equal to -1. d) The dynamical behavior of c) for the parallel update schedule. Note in this case that the network has no alliance.

EXAMPLE 2: reduction of the network from an alliance

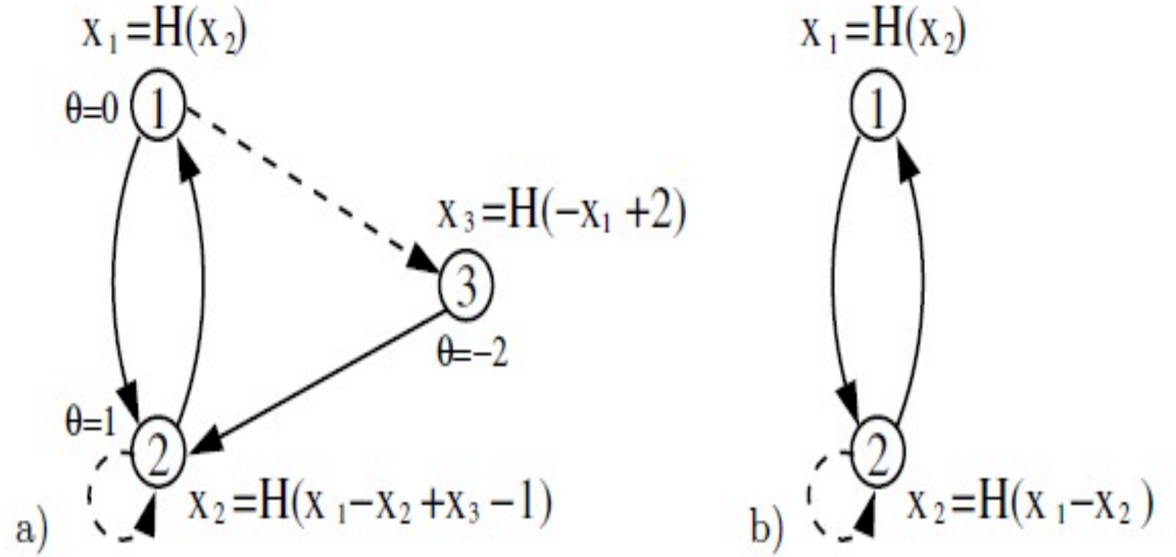


Fig. 16 a) A BN of 3 nodes with threshold Boolean functions defined as x_1, x_2, x_3 and thresholds $\theta = 0$, $\theta = 1$ and $\theta = -2$ respectively. Dotted lines represent negative interactions (weight= -1) while the others are positive interactions (weight= 1). b) The BN of a) is reduced after considering the alliance $x_3 = 1$. Note that, the effect of $x_3 = 1$ over x_2 is equivalent to write $x_2 = H(x_1 - x_2)$ while x_1 remains the same.

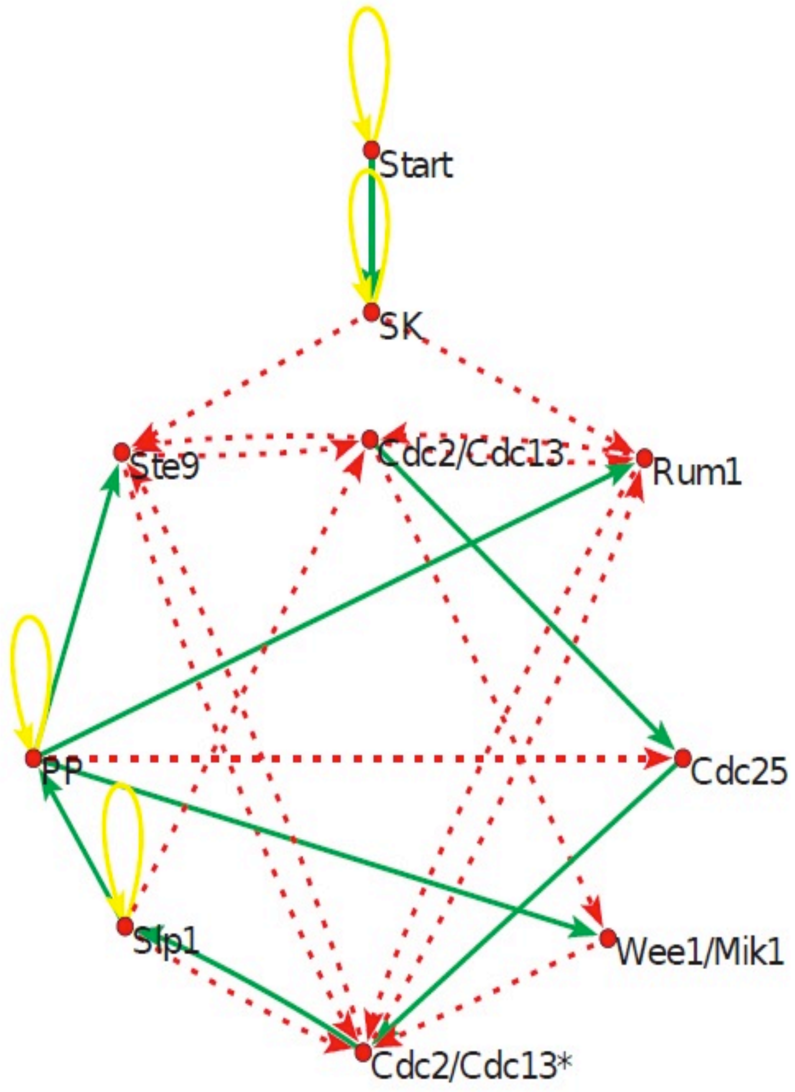


Fig. 3 The fission yeast cell-cycle threshold Boolean network. Using the same configuration as [3], (color online) the green/solid edges represent positive weights (activations), the red/dashed edges represent negative weights (inhibitory). The yellow/solid loops represent self-degradation.

Fission yeast cell-cycle model (*Yeast1)*

Model proposed in Davidich, Bornholdt (2008) PloSONE)

Threshold functions and interaction matrix for *Yeast1*

$$x_i' = H\left(\sum_{j=1}^{n} w_{ij}x_j - \theta_i\right) = \begin{cases} 0 & if \quad \sum_{j=1}^{n} w_{ij}x_j - \theta_i < 0 \\ 1 & if \quad \sum_{j=1}^{n} w_{ij}x_j - \theta_i > 0 \\ x_i & if \quad \sum_{j=1}^{n} w_{ij}x_j - \theta_i = 0 \end{cases}$$

$$W = \begin{array}{r} \\ Start \\ Sk \\ Cdc2/Cdc13 \\ Ste9 \\ Rum1 \\ Slp1 \\ Cdc2/Cdc13^{*} \\ Wee1/Mik1 \\ Cdc25 \\ PP \end{array} \begin{array}{c} \begin{array}{cccccccccc} Start & SK & Cdc2/Cdc13 & Ste9 & Rum1 & Slp1 & Cdc2/Cdc13^{*} & Wee1/Mik1 & Cdc25 & PP \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \end{array} \quad \Theta = \begin{pmatrix} 0 \\ 0 \\ -0.5 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Fig. 2 Weight matrix and threshold vector for *Yeast1*

Fixed points and the limit cycle (synchronous update)

Table 1 Attractors of *Yeast1* under the parallel updating mode. There are two types of attractors, fixed points (FP) and limit cycles (LC).

Attractor	Type	Basin size	*Start*	*SK*	*Cdc2/Cdc13*	*Ste9*	*Rum1*	*Slp1*	*Cdc2/Cdc13**	*Wee1/Mik1*	*Cdc25*	*PP*
1	FP	762	0	0	0	1	1	0	0	1	0	0
2	LC	208	0	0	0	0	0	0	0	0	1	1
	LC	0	0	0	0	0	0	1	0	0	1	0
	LC	0	0	0	1	1	1	0	1	1	0	0
3	FP	18	0	0	0	0	1	0	0	1	0	0
4	FP	18	0	0	0	1	0	0	0	1	0	0
5	FP	2	0	0	0	1	0	0	0	0	0	0
6	FP	2	0	0	0	1	0	0	0	0	1	0
7	FP	2	0	0	0	1	0	0	0	1	1	0
8	FP	2	0	0	0	0	1	0	0	0	0	0
9	FP	2	0	0	0	0	1	0	0	0	1	0
10	FP	2	0	0	0	0	1	0	0	1	1	0
11	FP	2	0	0	0	1	1	0	0	0	0	0
12	FP	2	0	0	0	1	1	0	0	0	1	0
13	FP	2	0	0	0	1	1	0	0	1	1	0

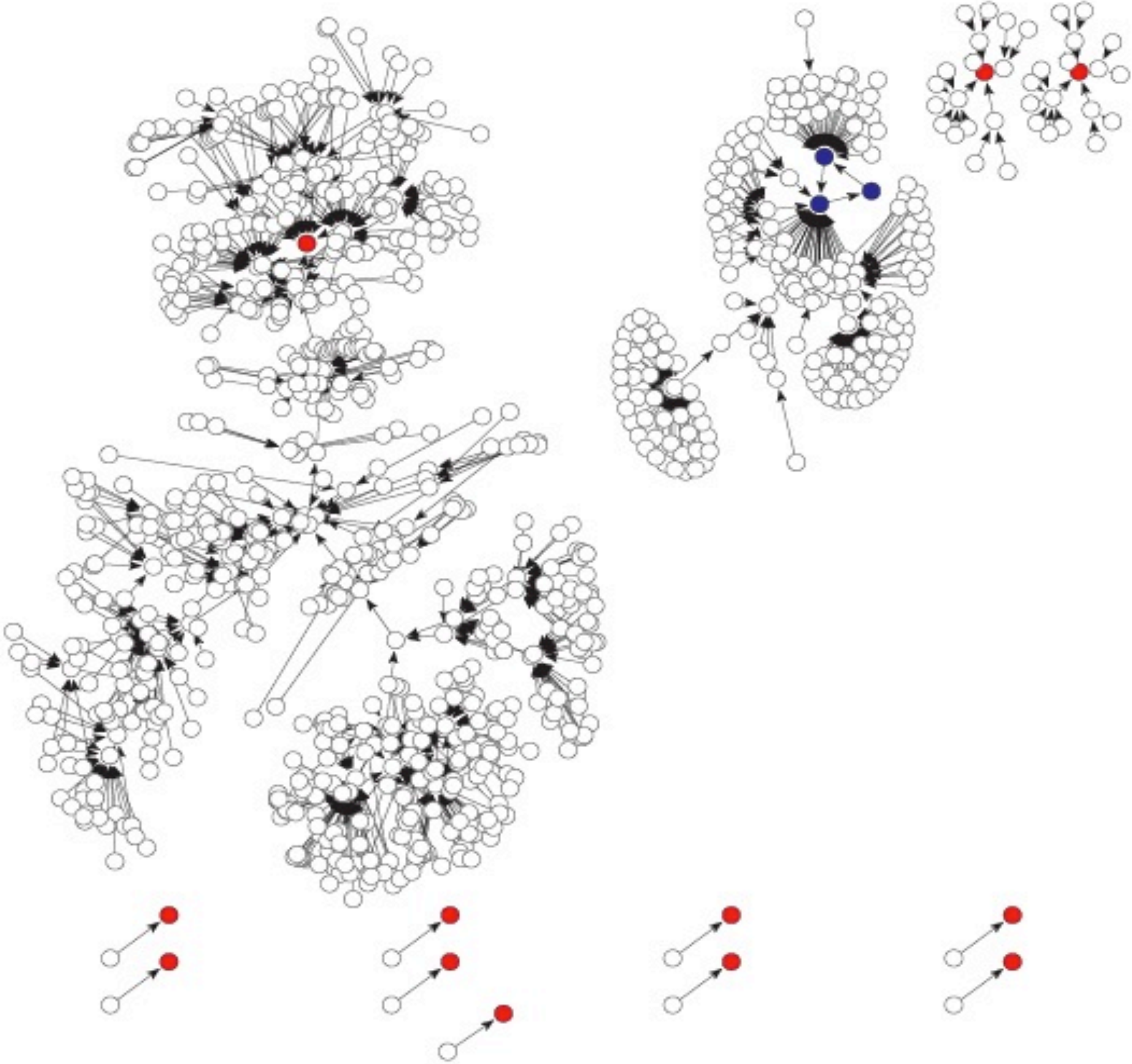
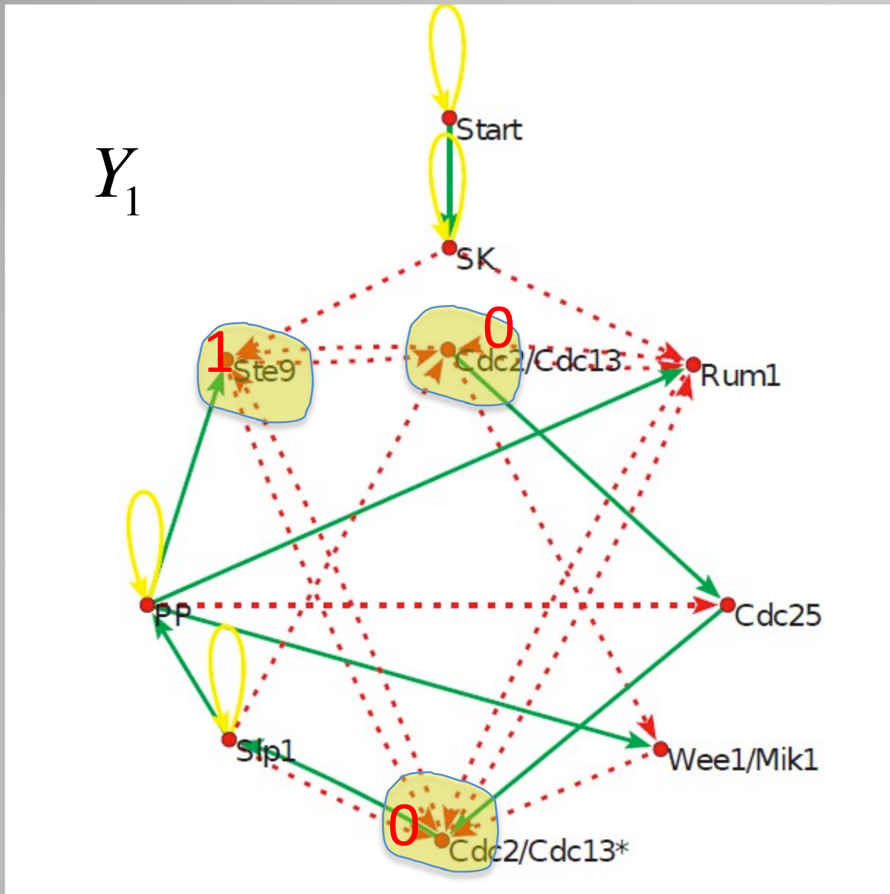
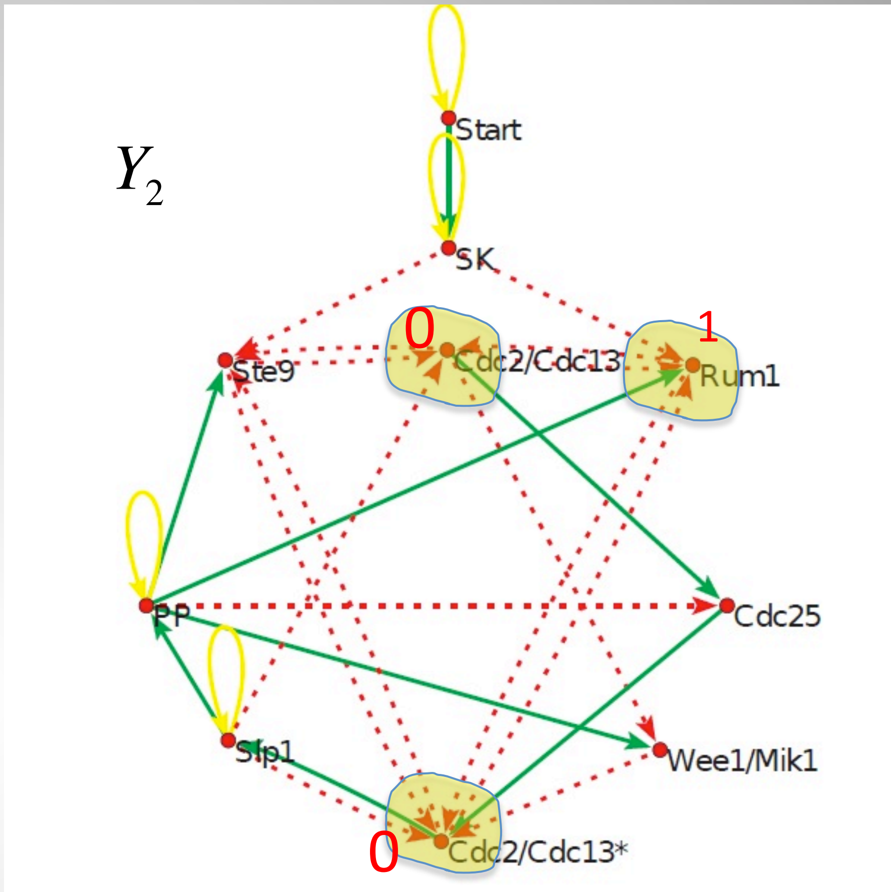


Fig. 7 State transition graph for *Yeast1* using the parallel updating scheme. (Color online) The twelve red circles represent the fixed point states, the three blue circles represent the states that belong to the limit cycle.





Alliances

$$Y_1 = (Cdc2/cdc13, Ste9, Cdc2/Cdc13^*) = (0,1,0)$$
$$Y_2 = (Cdc2/cdc13, Rum1, Cdc2/Cdc13^*) = (0,1,0)$$

For any initial vector such that it contains one of the two previous alliances then for any update mode the dynamics converges to fixed points

It remains to study what happens when in initial condition we do not have (as a sub-configuration) an *alliance*.

$$S_n = P_1(z) \cup P_2(z) \cup P_3(z) \cup P_4(z) \cup P_5(z) \cup P_6(z) \cup P_7(z) \cup P_8(z) \cup P_9(z)$$

$$z \in \{Ste9, Rum1\}$$

P_1	Cdc2/Cdc13	>	Z	<	Cdc2/Cdc13*
P_2	Cdc2/Cdc13	<	Z	>	Cdc2/Cdc13*
P_3	Cdc2/Cdc13	<	Z	<	Cdc2/Cdc13*
P_4	Cdc2/Cdc13	>	Z	>	Cdc2/Cdc13*
P_5	Cdc2/Cdc13	=	Z	<	Cdc2/Cdc13*
P_6	Cdc2/Cdc13	=	Z	>	Cdc2/Cdc13*
P_7	Cdc2/Cdc13	>	Z	=	Cdc2/Cdc13*
P_8	Cdc2/Cdc13	<	Z	=	Cdc2/Cdc13*
P_9	Cdc2/Cdc13	=	Z	=	Cdc2/Cdc13*

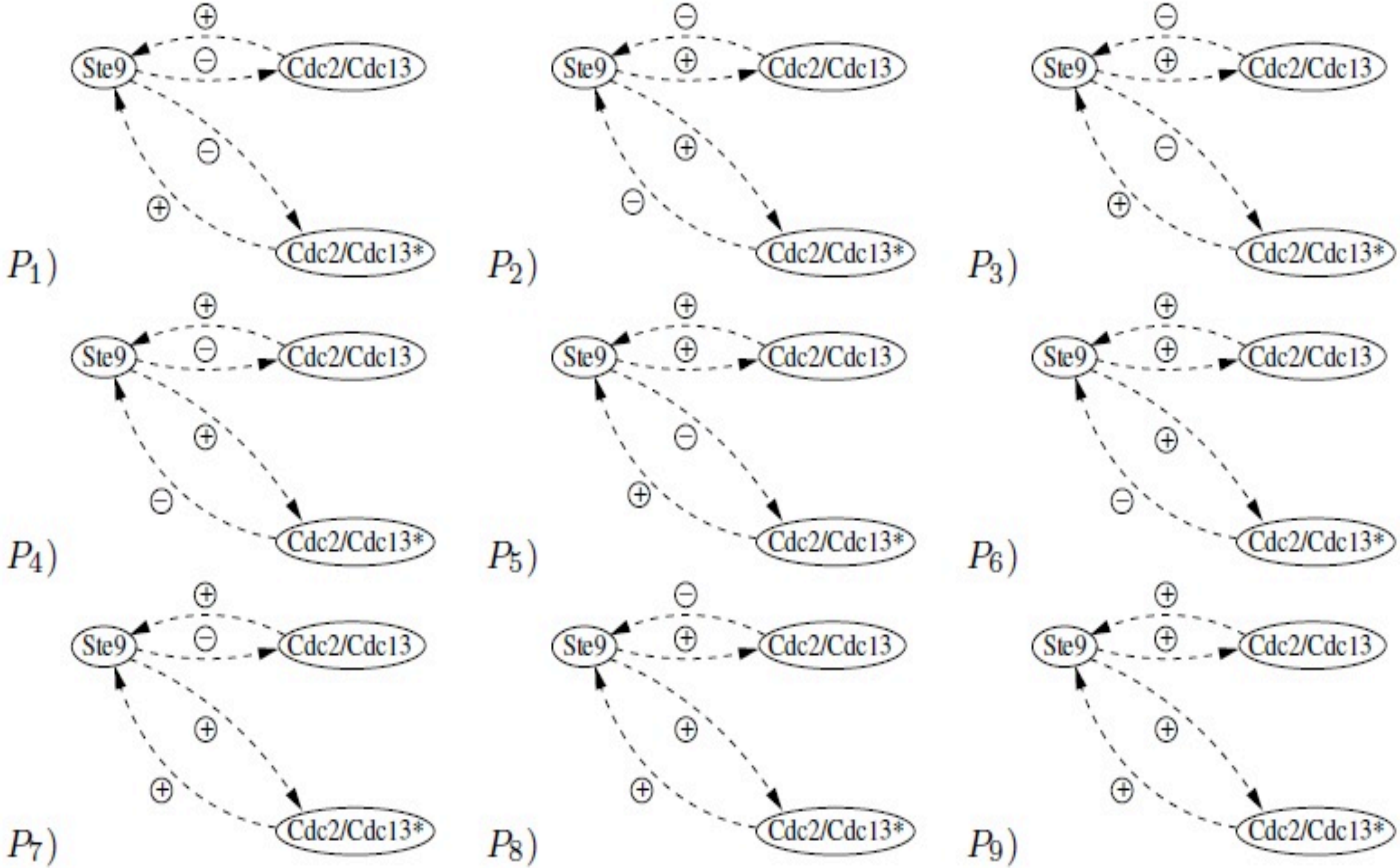


Fig. 5 The update subdigraphs associated with the partitions $P_1(Ste9)$,...,$P_9(Ste9)$ respectively, where an edge (i, j) is labeled by $\oplus$ if and only if $s(i) \geq s(j)$, otherwise, the edge is labeled by $\ominus$ (see [17] for more details)

The total number of updates is 545835

Bby algoritmhs and the notion of alliances, we reduce to study only 15350 diffferent equivalent classes

So there exists 5513 classes with a limit cycle (period between 2 and 5)

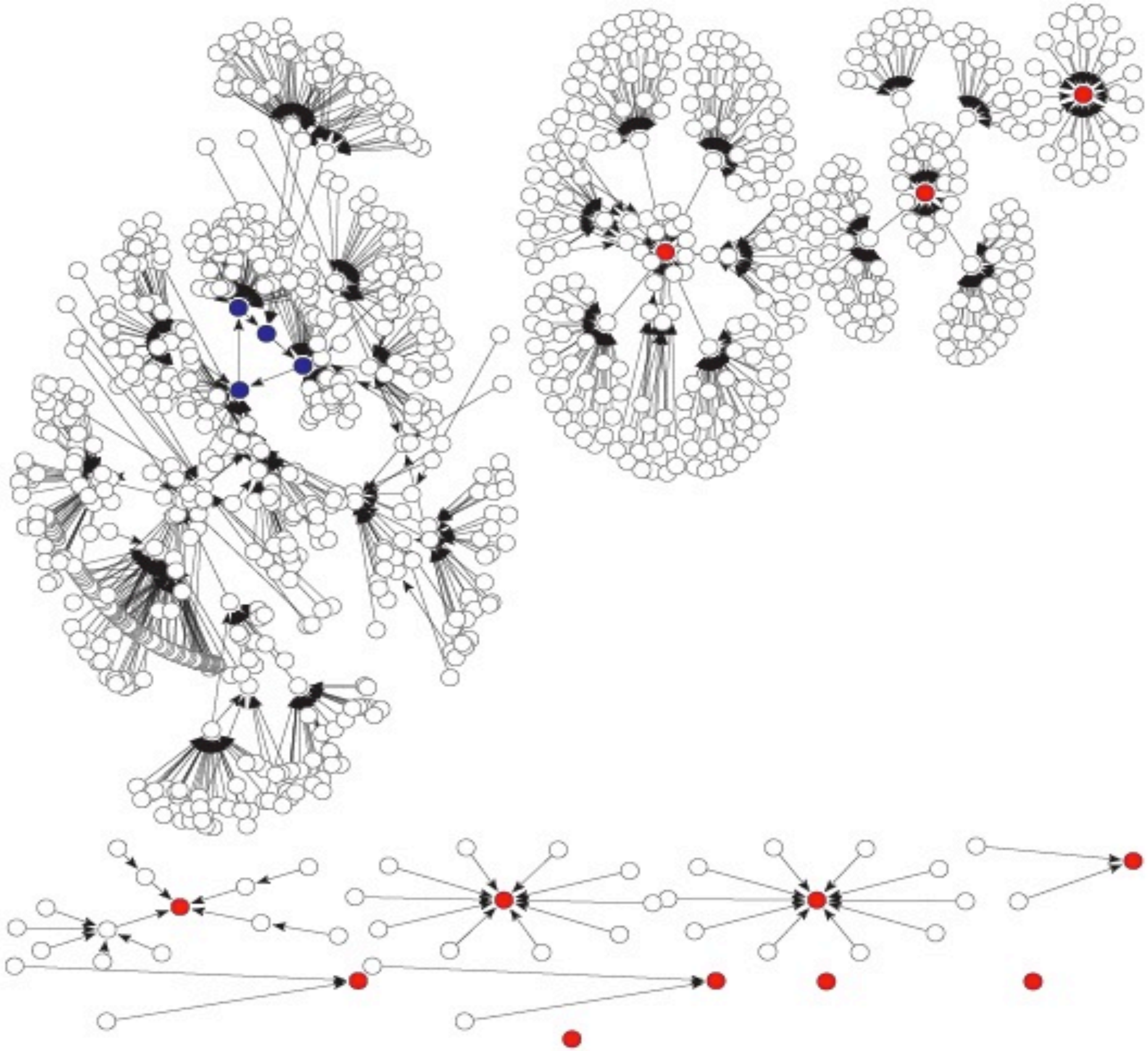


Fig. 8 State transition graph for *Yeast1* using the block-sequential updating mode: $s(Wee1/Mik1) = s(Cdc25) < s(Cdc2/Cdc13) = s(Cdc2/Cdc13^*) = s(Ste9) < s(Rum1) = s(Slp1) = s(PP) = s(Start) = s(SK)$. (Color online) The twelve red circles represent the fixed point states, the four blue circles represent the states that belong to the limit cycle.

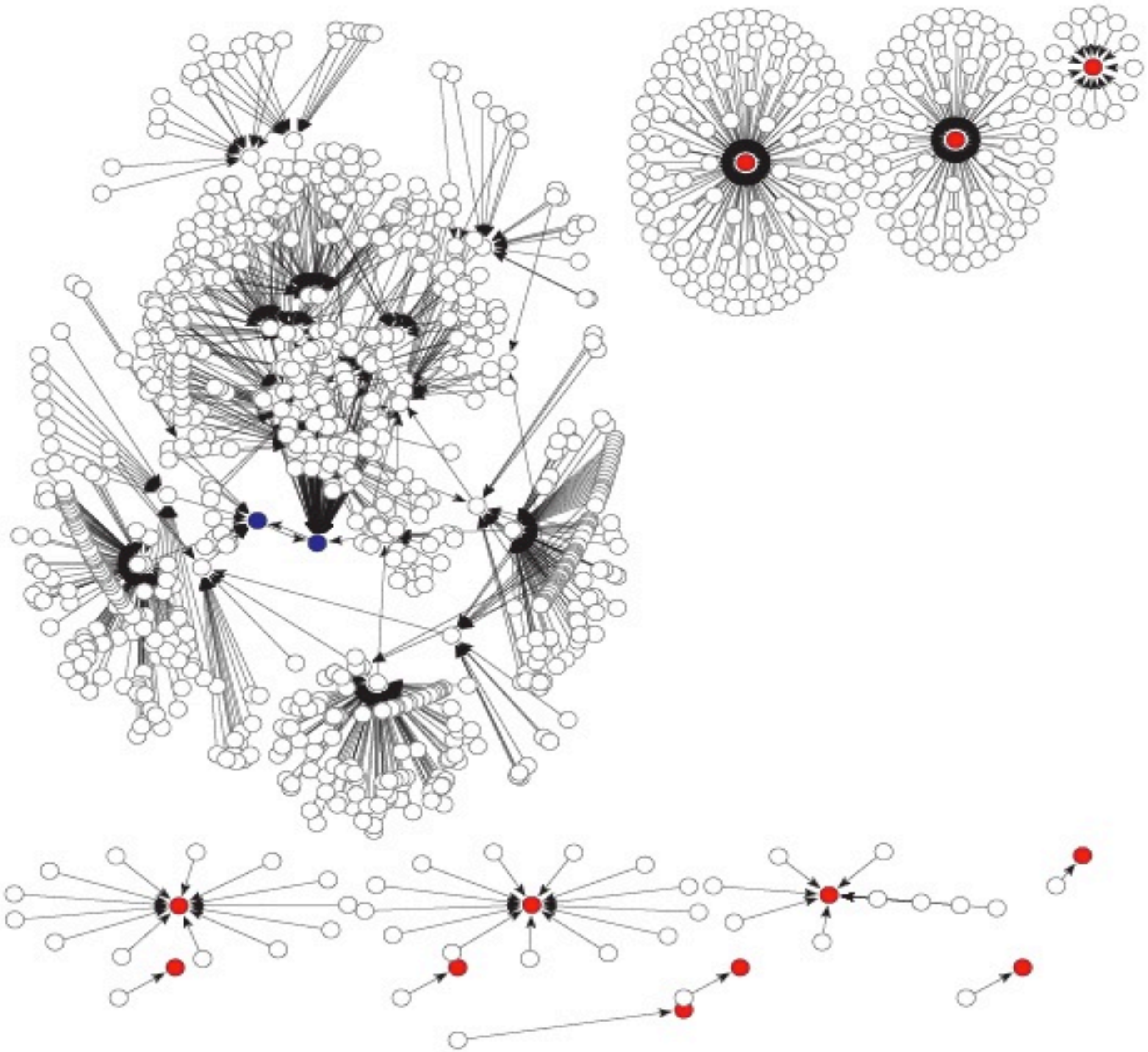


Fig. 9 State transition graph for *Yeast1* using the block-sequential updating mode: $s(Rum1) < s(Cdc2/Cdc13^*) < s(Slp1) < s(Cdc2/Cdc13) < s(Ste9) = s(Wee1/Mik1) = s(Cdc25) = s(PP) < s(Start) = s(SK)$. (Color online) The twelve red circles represent the fixed point states, the two blue circles represent the states that belong to the limit cycle.

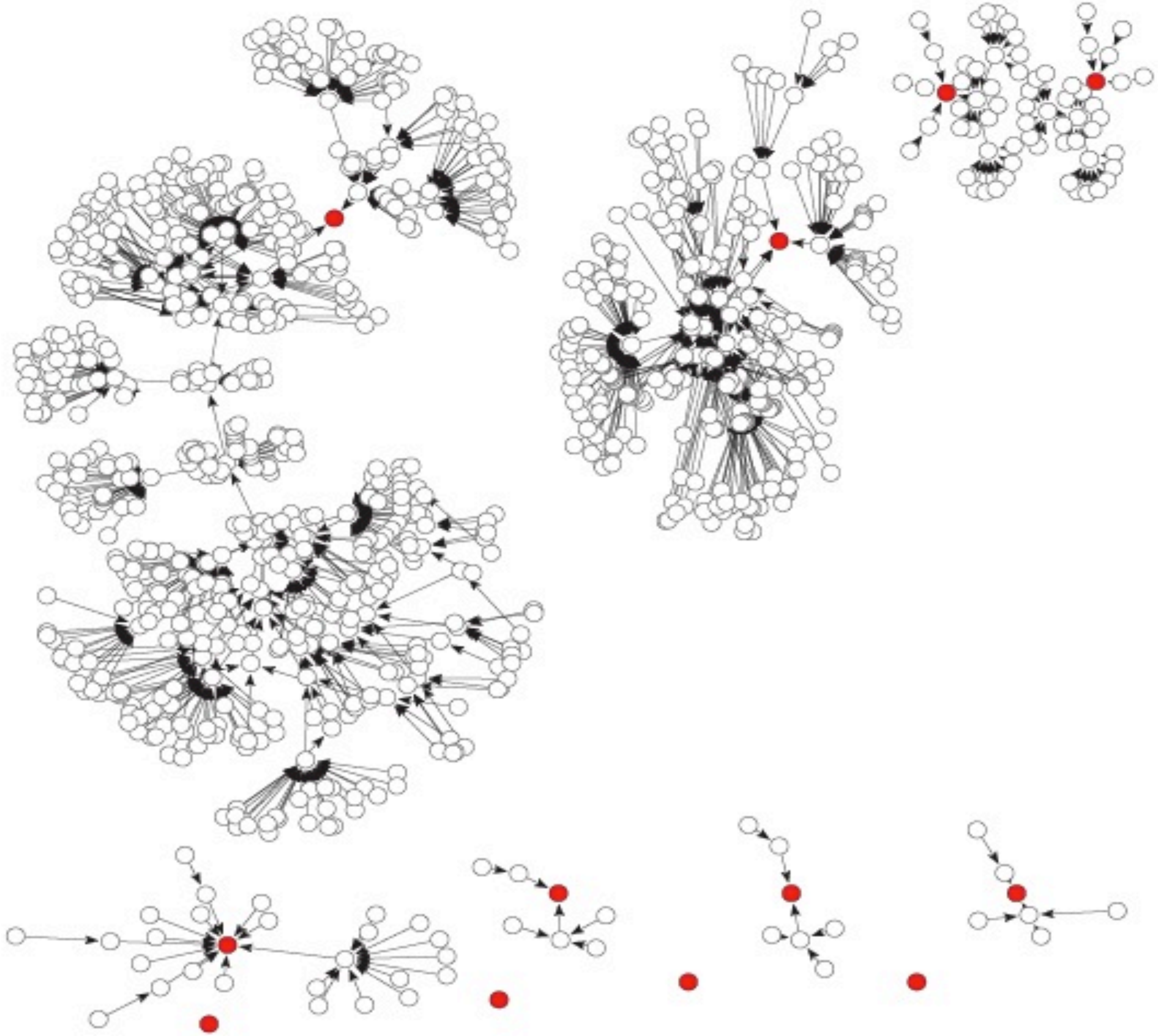


Fig. 10 State transition graph for *Yeast1* using the block-sequential updating mode: $s(Ste9) < s(PP) < s(Wee1) = s(Mik1) = s(Cdc25) < s(Start) = s(SK) = s(Cdc2/Cdc13) = s(Rum1) = s(Slp1) = s(Cdc2/Cdc13^{*})$. (Color online) The twelve red circles represent the fixed point states, there are no limit cycles.

Cell cycle of the budding yeast

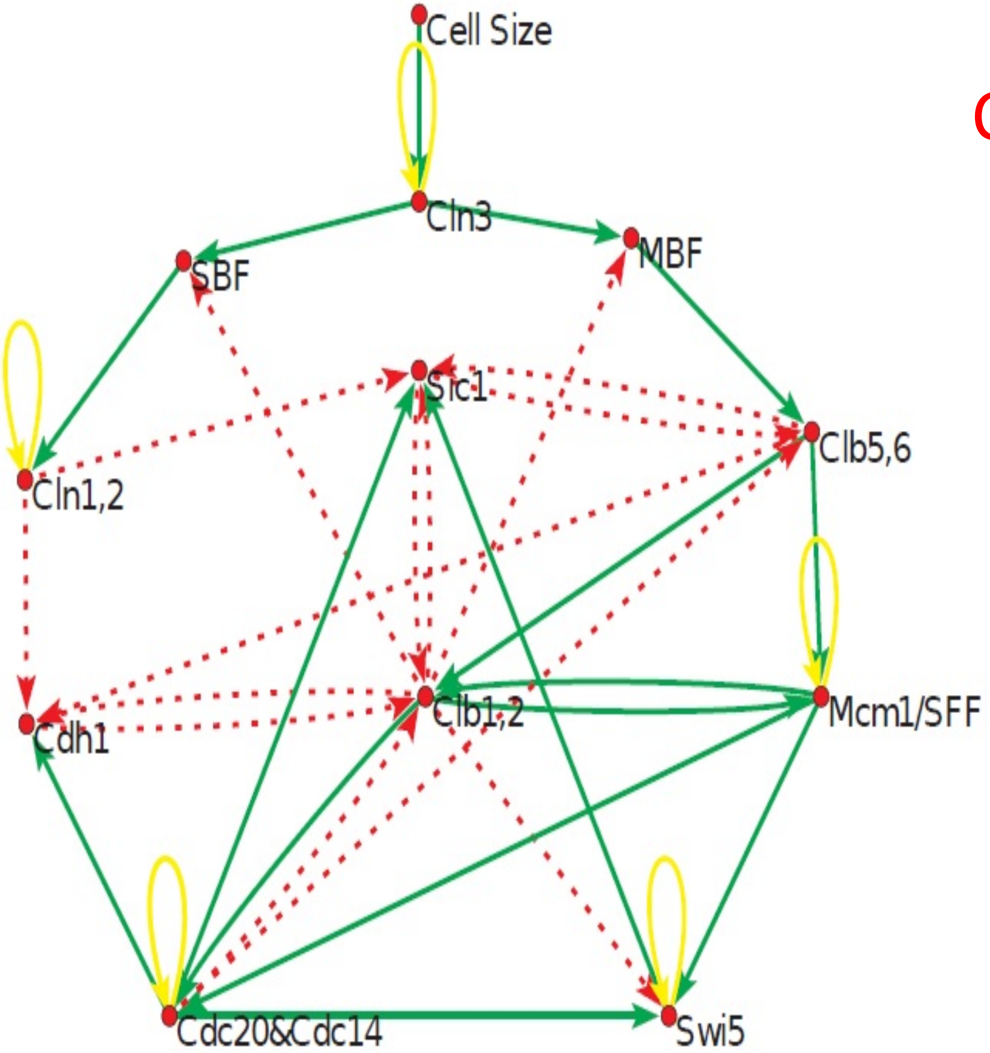


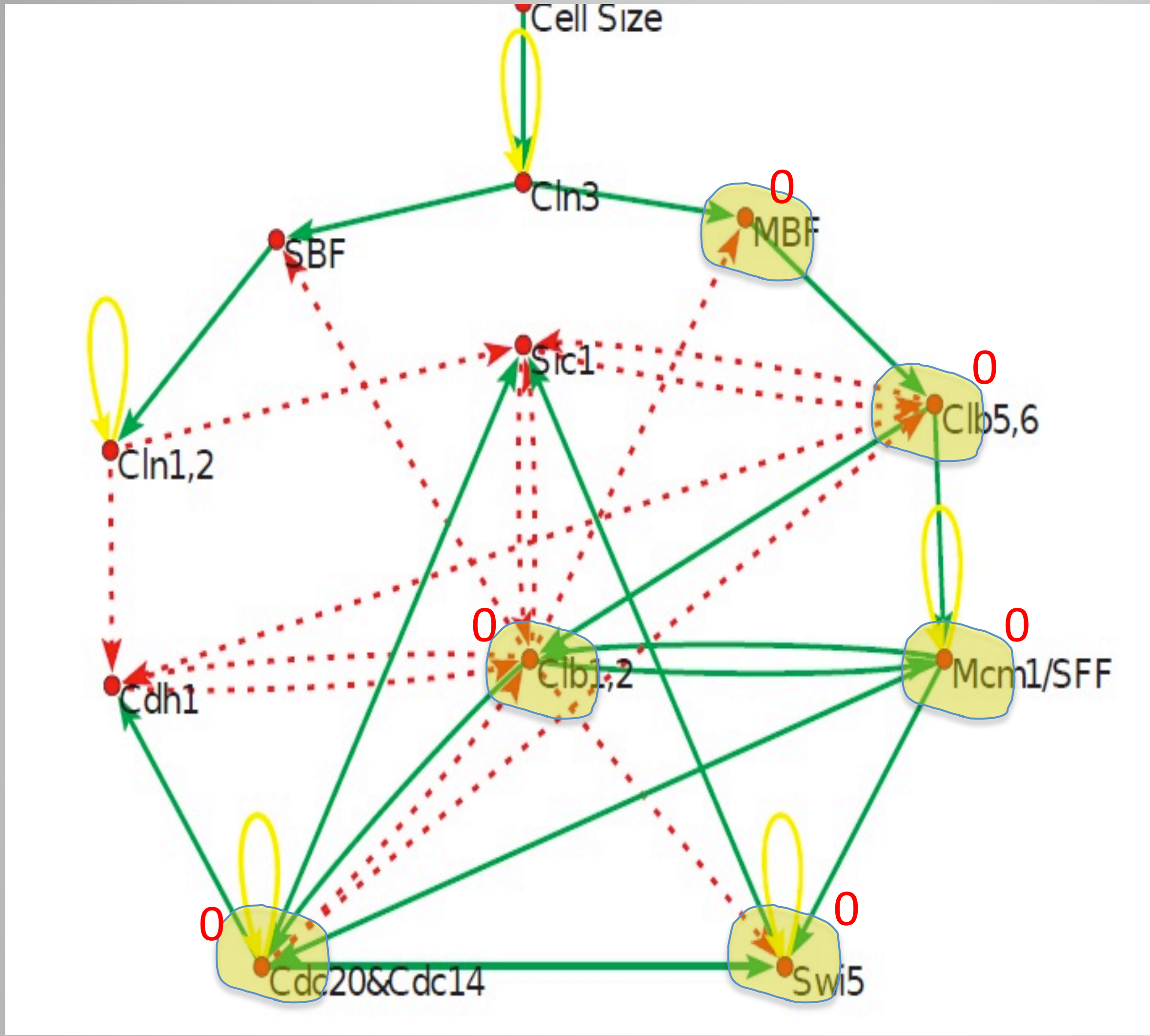
Fig. 12 The budding yeast cell-cycle threshold Boolean network. Using the same configuration as [14], (color online) the green/solid edges represent positive weights (activations), the red/dashed edges represent negative weights (inhibitory). The yellow/solid loops represent self-degradation.

Li, Long, Lu, Tang,(2004) The yeast cell-cycle network is robustly designed, PNAS,101, 4781-4786

Matrix and transition vector of the yeast2 network

$$W = \begin{array}{cc}
 & \begin{array}{ccccccccccc} Cln3 & MBF & SBF & Cln1,2 & Cdh1 & Swi5 & Cdc20 & Clb5,6 & Sic1 & Clb1,2 & Mcm1 \end{array} \\
\begin{array}{c} Cln3 \\ MBF \\ SBF \\ Cln1,2 \\ Cdh1 \\ Swi5 \\ Cdc20 \\ Clb5,6 \\ Sic1 \\ Clb1,2 \\ Mcm1 \end{array} &
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1
\end{pmatrix}
\end{array}
\quad \Theta = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Fig. 11 Weight matrix and threshold vector for *Yeast2*

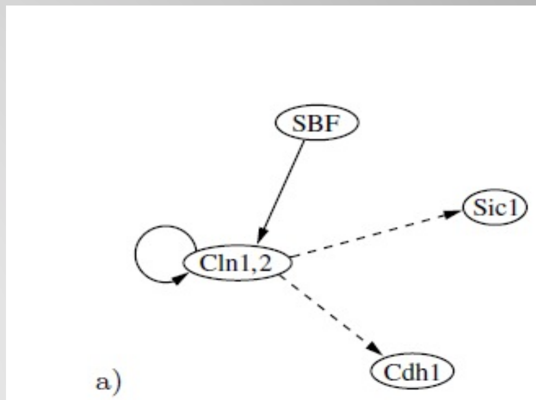


0 *Alliance elements*

For any update *yeast2* admits only fixed points

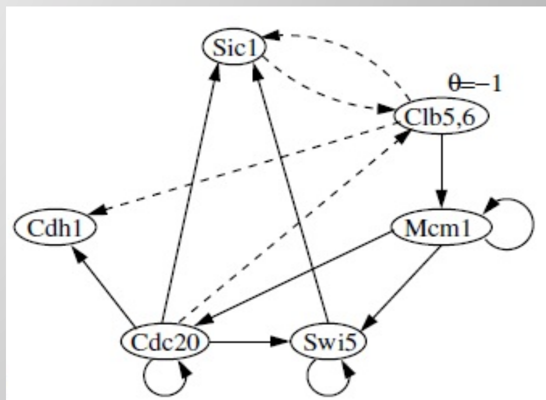
To prove that it's enough to study two cases: MBF= 0 or MBF=1

For MBF=0, any initial configuration converges to the alliance. After that, the sub-network to analyze is a tree, so for any initial vector and any update it converges to fixed points.



Tree for MBF=0

For MBF =1 by studying numerically the equivalent classes for a reduced network, we determine that every dynamics admits only fixed points.



Reduced network for MBF=1

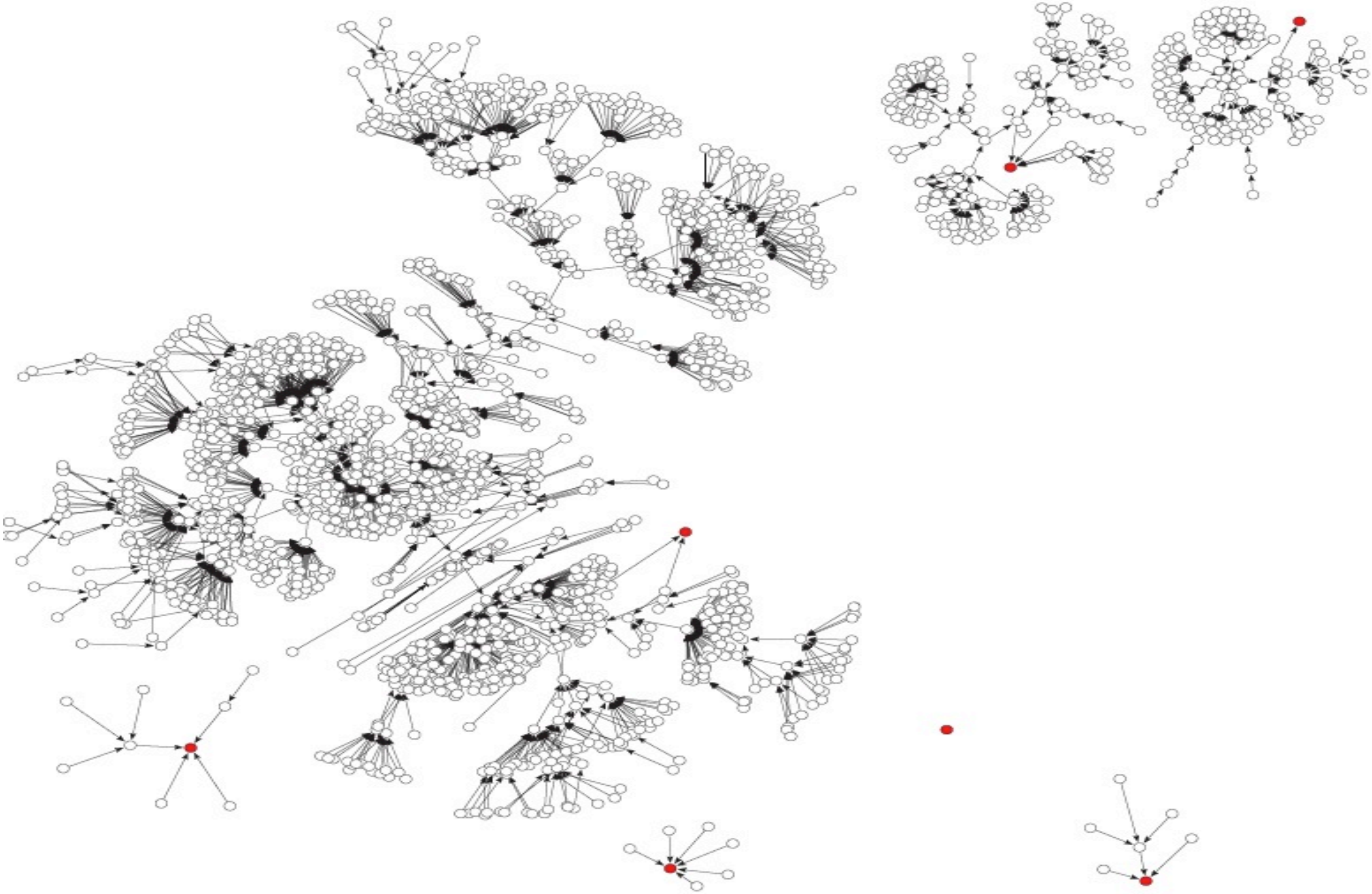


Fig. 14 State transition graph for *Yeast2* using the parallel updating scheme. (Color online) The seven red circles represent the fixed point states, there are no limit cycles.

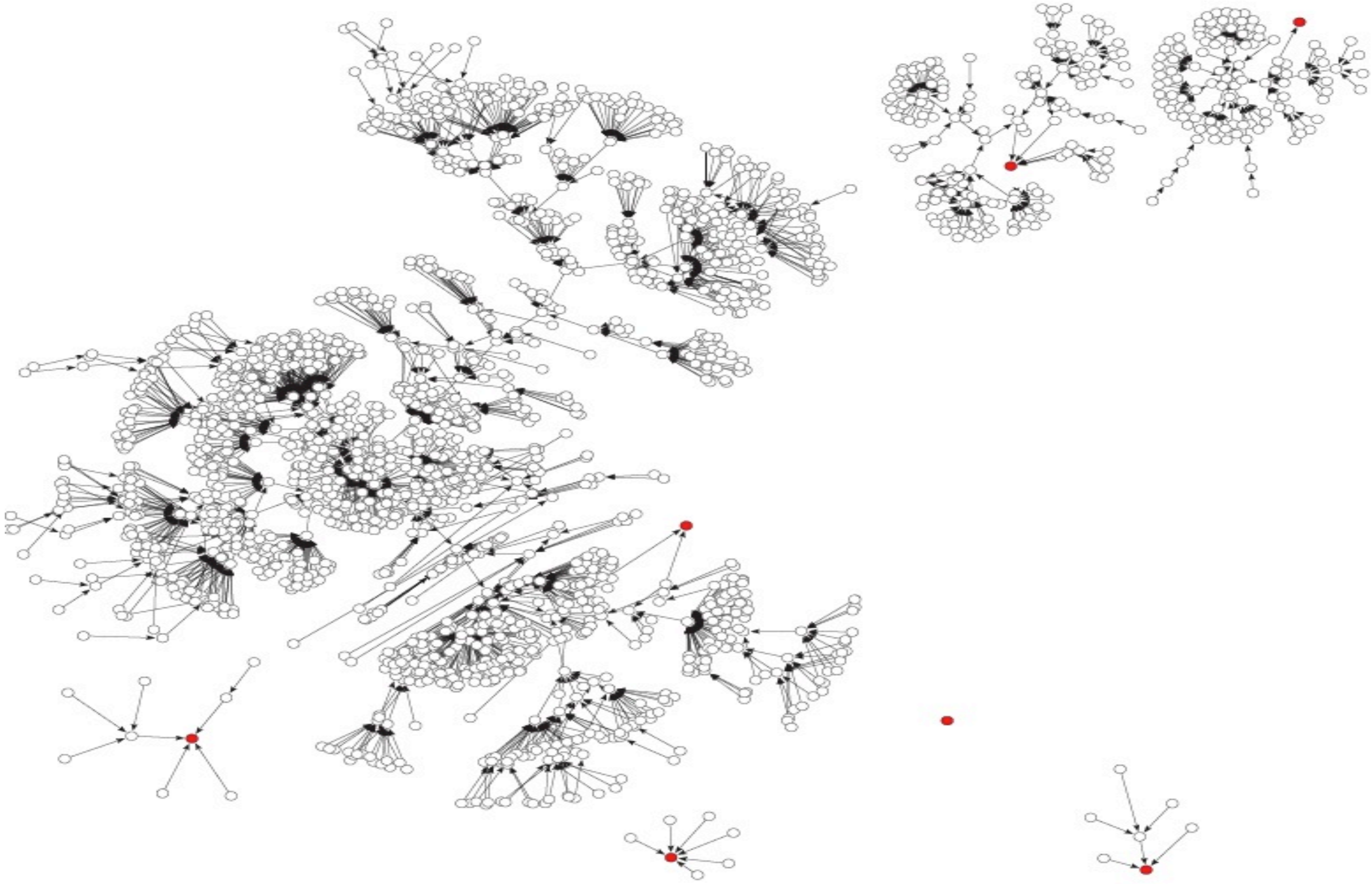


Fig. 14 State transition graph for *Yeast2* using the parallel updating scheme. (Color online) The seven red circles represent the fixed point states, there are no limit cycles.

Gracias !!!