On the extension complexity of polytopes

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A polytope $P$ in $\mathbb{R}^d$ is the convex hull of finitely many points. Its faces are intersections with supporting hyperplanes. Faces of dimensions 0 and $d - 1$ are vertices and facets, respectively. 

An extended formulation of $P$ is a polytope that can be linearly projected onto $P$. 

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The extension complexity $xc(P)$ is the minimal number of facets of an extended formulation.

$$ \dim + 1 \leq xc(P) \leq \min(\# \text{ vertices } P, \# \text{ facets } P) $$
Motivation: linear programming

If \( \pi(Q) = P \), and \( Q \) has much fewer facets than \( P \), then it is more **efficient** to optimize over \( Q \) than over \( P \)... 

- Swart 80’s claims that \( P = NP \) because the TSP-polytope (associated to the *Travelling Salesman Problem*) has a polynomial size extension

- Yannakakis 1991: Every **symmetric** extended formulation of TSP-polytope has exponential size (**so Swart’s proof was wrong**)

- Kaibel, Pashkovich and Theis 2010: symmetry matters

- Fiorini, Massar, Pokutta, Tiwary and de Wolf 2015: the extension complexity of the TSP-polytope is exponential
The \textit{nonnegative rank} of the nonnegative $n \times m$ matrix $S$ is the smallest $k$ such that there are nonnegative $n \times k$ and $k \times m$ matrices $A$ and $B$ such that

$$S = A \cdot B$$

\[
\begin{bmatrix}
0 & 0 & 1 & 2 & 2 & 1 \\
1 & 0 & 0 & 1 & 2 & 2 \\
2 & 1 & 0 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 & 0 & 1 \\
1 & 2 & 2 & 1 & 0 & 0 \\
0 & 1 & 2 & 2 & 1 & 0
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
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\end{bmatrix}
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0 & 0 & 1 & 1 & 0 & 0 \\
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1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 & 1 \\
1 & 2 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
Theorem (Yannakakis ’91)

The extension complexity of $P$ coincides with the nonnegative rank of $S(P)$.

Let $v_1, \ldots, v_n$ be the vertices of $P$, $f_1, \ldots, f_m$ its facets, with $f_j$ supported by $\langle a_j, x \rangle = b_j$. The slack matrix of $P$ is the $n \times m$ matrix $S(P)$ with entries

$$S_{ij} = \langle a_j, v_i \rangle = b_j$$

$$S(P) = \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$
Polygons
The extension complexity of a hexagon $\diamond$ is either 5 or 6 . . .

- Every $\geq 5$-polytope has $\geq 6$ facets
- The only 4-polytope with 5 facets is the 4-simplex, its projections have $\leq 5$ vertices
- The only 3-polytope with $\leq 5$ facets and $\geq 6$ vertices is the *triangular prism*
Warm-up: Hexagons

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Regular n-gons

Theorem (Ben-Tal-Nemirovski 2001)

Let $P$ be a regular $n$-gon, then $xc(P) = \Theta(\log(n))$
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If $P = \pi(Q)$, the preimage of each face of $P$ is a face of $Q$. A polytope with $m$ facets has $\leq 2^m$ faces.
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**Worst-case polygons**

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Let’s try heptagons...
Worst-case polygons

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*Are there, for each \( n \geq 3\), \( n\)-gons with extension complexity \( n\)?*

Let’s try heptagons...
Worst-case heptagons

Can we do this for every heptagon?

No... but at least for some rotation?
Worst-case heptagons

Yes! Otherwise, we get a realization of this *non-stretchable* arrangement!
Theorem (Shitov 2013, P.-Pfeifle 2014)

Every heptagon has extension complexity 6.
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*Every heptagon has extension complexity 6.*

Corollary

*Any n-gon P with n ≥ 7 is a projection of a *(2 + ⌊ n/7 ⌋)*-dimensional polytope with at most ⌊ 6n/7 ⌋ facets. Hence,

\[ xc(P) \leq \left\lfloor \frac{6n}{7} \right\rfloor \]
Worst-case heptagons

Theorem (Shitov 2013, P.-Pfeifle 2014)
Every heptagon has extension complexity 6.

Corollary
Any $n$-gon $P$ with $n \geq 7$ is a projection of a $(2 + \lceil \frac{n}{7} \rceil)$-dimensional polytope with at most $\left\lceil \frac{6n}{7} \right\rceil$ facets. Hence,

$$xc(P) \leq \left\lceil \frac{6n}{7} \right\rceil$$

Theorem (Shitov 2015)
If $P$ is an $n$-gon with $n$ large enough, then $xc(P) \leq \frac{25n}{\sqrt{\ln \ln \ln \ln \ln n}}$
Polytopes with few vertices or facets
Goal: Study extension complexity of $d$-polytopes with few vertices/facets ($d + 1 + \alpha$ for fixed (small) $\alpha$).
In particular, $d$-polytopes with $d + 4$ vertices/facets

Motivation 1: Produce high-dimensional (interesting) examples. $d$-polytopes with $d + 4$ vertices are Sturmfels’s “threshold for counterexamples”:
Many combinatorial types, intricate realization spaces

Motivation 2: If there was an $\alpha$ such that each $d$-polytope with $d + 1 + \alpha$ vertices has $xc \leq d + \alpha$, it would provide non-trivial upper bounds for $xc$ with respect to the number of vertices...
A generic $d$-polytope $P$ with $d + 1 + \alpha$ facets has extension complexity

$$xc(P) \geq 2\sqrt{d(n - 1)} - d + 1$$
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$D$-polytopes with $N$ facets
$$\dim = N \cdot D$$

$\pi^{-1}(P)$
$$\dim \geq (D + 1)(D - d)$$

$d$-polytopes with $n$ facets
$$\dim = n \cdot d$$

$\pi(P)$
$$\dim \leq DN - (D + 1)(D - d)$$

$\Rightarrow$ A generic $P$ has $xc(P) \geq \min_D \frac{(D+1)(D-d)+nd}{D} = 2 \sqrt{nd-d-d+1}$
### Corollary

For any $d > \left(\frac{\alpha - 1}{2}\right)^2$ there are $d$-polytopes with $d + 1 + \alpha$ vertices/facets with extension complexity $d + 1 + \alpha$
Theorem (P. 2016)

Let $P$ be a $d$-polytope with $d + 4$ vertices, then

1. $xc(P) = d + 2$ if and only if $P$ has $d + 2$ facets.

2. $xc(P) = d + 3$ if and only if:
   1. $P$ has $d + 3$ facets, or
   2. $P = \pi(Q)$, where $Q \cong \text{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ for some affine projection $\pi$. In this case, either
      1. $P = \text{pyr}_k(Q)$ where $Q$ is a Desarguian hexagon (a hexagon with $xc(Q) = 5$), or
      2. $P$ has a subset of 6 vertices forming a triangular prism.

3. $xc(P) = d + 4$ otherwise.
**Theorem (P. 2016)**

Let $P$ be a $d$-polytope with $d + 4$ vertices, then

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   **Combinatorial condition, 1 combinatorial type**

2. $xc(P) = d + 3$ if and only if:
   2.1 $P$ has $d + 3$ facets, or
   **Combinatorial condition, $\leq 8$ combinatorial types**
   2.2 $P = \pi(Q)$, where $Q \cong \text{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ for some affine projection $\pi$.
   **In this case, either**
   2.2.1 $P = \text{pyr}_k(Q)$ where $Q$ is a Desarguian hexagon (a hexagon with $xc(Q) = 5$), or
   **Geometric condition, 1 combinatorial type**
   2.2.2 $P$ has a subset of 6 vertices forming a triangular prism.
   **Geometric condition, $\theta(d^2)$ combinatorial types**

3. $xc(P) = d + 4$ otherwise.
   **Superexponentially many, $\geq d^{(1/2-o(1))}$ combinatorial types**
There are few polytopes with few vertices and few facets

**Theorem (P. 2016)**

\[ \exists D(\alpha, \beta) \text{ such that every } d\text{-polytope with } \leq d + 1 + \alpha \text{ vertices, } \leq d + 1 + \alpha \text{ facets and } d > D(\alpha, \beta) \text{ is a pyramid.} \]
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Corollary

For fixed \( \alpha, \beta, \) the number of combinatorial types of \( d\)-polytopes with \( \leq d + 1 + \alpha \) vertices and \( \leq d + 1 + \alpha \) facets is bounded by a constant independent of \( d \).
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**Proof.**
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∃ $D(\alpha, \beta)$ such that every $d$-polytope with $\leq d + 1 + \alpha$ vertices, $\leq d + 1 + \alpha$ facets and $d > D(\alpha, \beta)$ is a pyramid.

Proof.

Theorem (Marcus 1981)

∃ $f(\alpha)$ such that for $d > f(\alpha)$ every $d$-polytope with $\leq d + 1 + \alpha$ vertices has a vertex adjacent to all the other vertices.

By induction on $\beta$. Since $d > D(\alpha, \beta) \geq f(\alpha)$, $\exists v$ adjacent to all the other vertices.

If there is only one facet non-adjacent to $v \Rightarrow P$ is a pyramid.
Otherwise, the vertex-figure $P/v$ has $(d - 1) + \alpha + 1$ vertices and $< (d - 1) + \beta + 1$ facets. By induction, $P/v$ is a pyramid $\Rightarrow P$ is a pyramid.
Non-pyramidal d-polytopes with d+4 vertices and d+3 facets
Non-pyramidal d-polytopes with d+4 vertices and d+3 facets

Merci beaucoup!