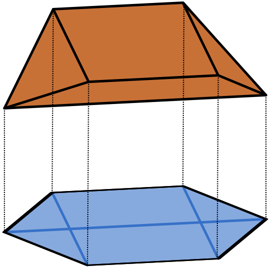

On the extension complexity of polytopes

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Journées nationales GDR IM 2017
15/03/2017

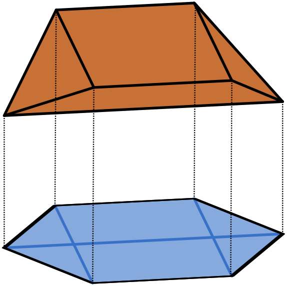
Extension complexity

A *polytope* P in $\mathbb{R}^d$ is the convex hull of finitely many points

Its *faces* are intersections with supporting hyperplanes. Faces of dimensions 0 and $d-1$ are *vertices* and *facets*, respectively

An *extended formulation* of P is a polytope that can be linarly projected onto P

The *extension complexity* $\mathrm{xc}(P)$ is the minimal number of facets of an extended formulation



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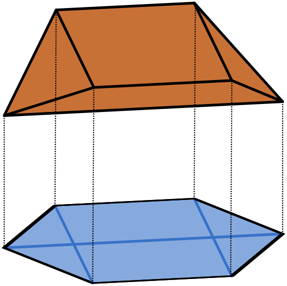


$\dim + 1 \leq \mathrm{xc}(P) \leq \min(\#\text{ vertices } P, \#\text{ facets } P)$

Motivation: linear programming

If $\pi(Q) = P$, and Q has much fewer facets than P, then it is more efficient to optimize over Q than over P...

- Swart 80's claims that $P = NP$ because the TSP-polytope (associated to the *Travelling Salesman Problem*) has a polynomial size extension
- Yannakakis 1991: Every symmetric extended formulation of TSP-polytope has exponential size (so Swart's proof was wrong)
- Kaibel, Pashkovich and Theis 2010: symmmetry matters
- Fiorini, Massar, Pokutta, Tiwary and de Wolf 2015: the extension complexity of the TSP-polytope is exponential

Nonnegative rank

The *nonnegative rank* of the nonnegative $n \times m$ matrix S is the smallest k such that there are nonnegative $n \times k$ and $k \times m$ matrices A and B such that

$$S = A \cdot B$$

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

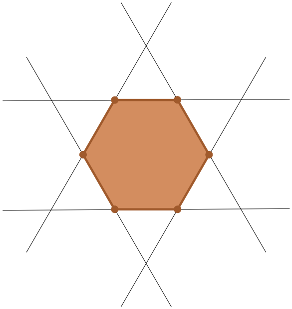
Nonnegative rank

Theorem (Yannakakis '91)

The extension complexity of P coincides with the nonnegative rank of $S(P)$

Let $v_1, \ldots, v_n$ be the vertices of P, $f_1, \ldots, f_m$ its facets, with f_j supported by $\langle a_j, x \rangle = b_j$. The *slack matrix* of P is the $n \times m$ matrix $S(P)$ with entries

$$S_{ij} = \langle a_j, v_i \rangle = b_j$$



$$S(P) = \begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

Polygons

Warm-up: Hexagons

The extension complexity of a hexagon ⬡ is either 5 or 6 . . .

- Every ≥ 5-polytope has ≥ 6 facets
- The only 4-polytope with 5 facets is the 4-simplex, its projections have ≤ 5 vertices
- The only 3-polytope with ≤ 5 facets and ≥ 6 vertices is the *triangular prism*

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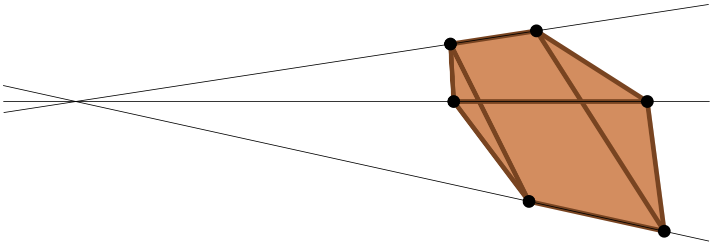
. . . and $\mathrm{xc}(⬡) = 5 \iff$ ⬡ is Desarguian



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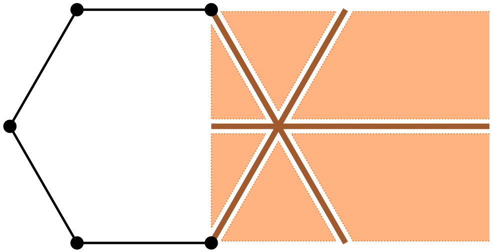

Regular n-gons

Theorem (Ben-Tal-Nemirovski 2001)

Let P be a regular n-gon, then $\mathrm{xc}(P) = \Theta(\log(n))$

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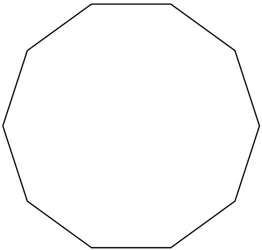

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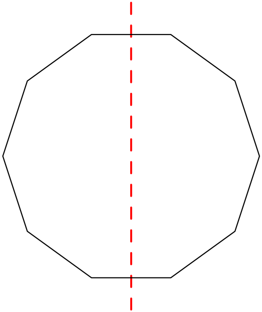

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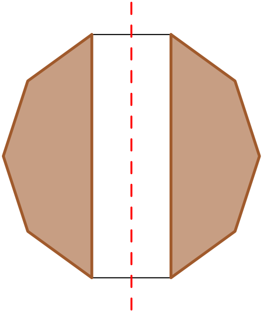

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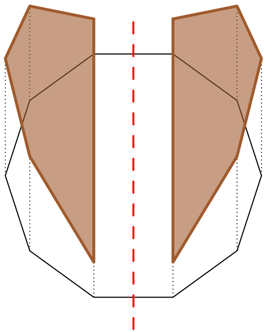

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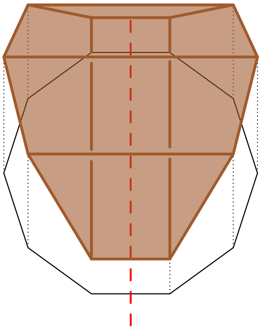

Worst-case polygons

Question (Beasley and Laffey 2009)

Are there, for each $n \geq 3$, n-gons with extension complexity n?

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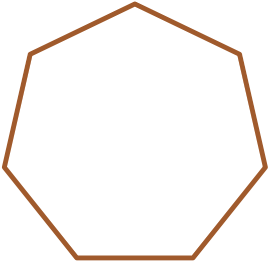

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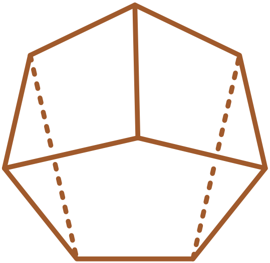
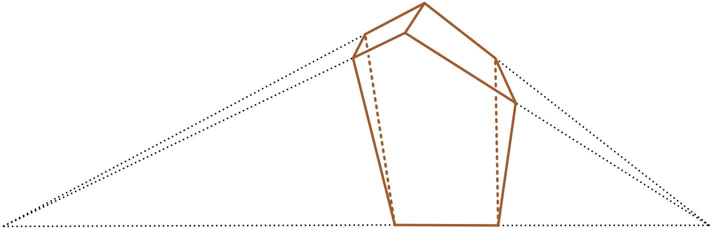
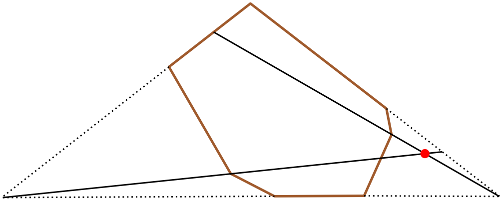


Worst-case heptagons

Can we do this for every heptagon?

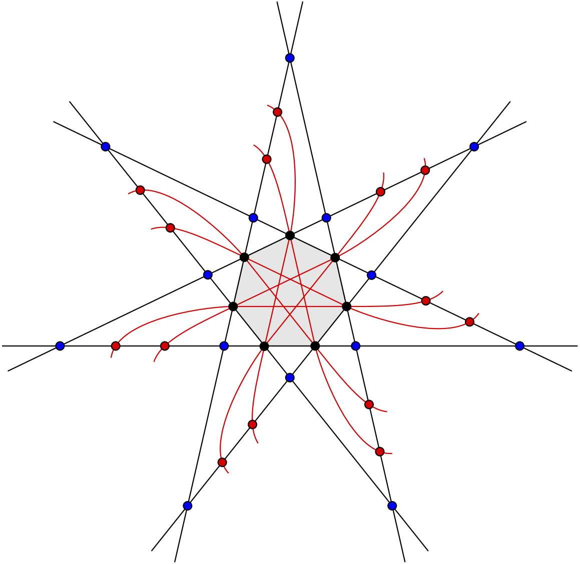




No... but at least for some rotation?

Worst-case heptagons

Yes! Otherwise, we get a realization of this *non-strechable* arrangement!



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Theorem (Shitov 2013, P.-Pfeifle 2014)

Every heptagon has extension complexity 6*.*

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Corollary

Any n-gon P with $n \geq 7$ *is a projection of a* $(2 + \lfloor \frac{n}{7} \rfloor)$*-dimensional polytope with at most* $\lceil \frac{6n}{7} \rceil$ *facets. Hence,*

$$\mathrm{xc}(P) \leq \left\lceil \frac{6n}{7} \right\rceil$$

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Theorem (Shitov 2015)

If P is an n-gon with n large enough, then $\mathrm{xc}(P) \leq \frac{25n}{\sqrt{\ln \ln \ln \ln \ln \ln n}}$

Polytopes with few vertices or facets

Polytopes with few vertices (or facets)

Goal: Study extension complexity of d-polytopes with *few* vertices/facets ($d+1+\alpha$ for fixed (small) α).
In particular, d-polytopes with $d+4$ vertices/facets

Motivation 1: Produce high-dimensional (interesting) examples.
d-polytopes with $d+4$ vertices are Sturmfels's "*threshold for counterexamples*":
Many combinatorial types, intricate realization spaces

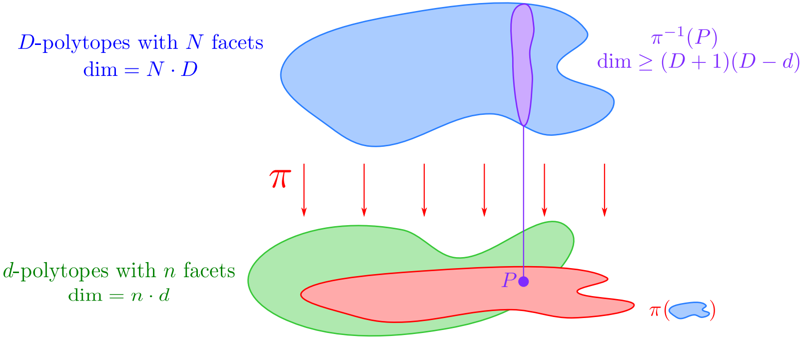
Motivation 2: If there was an α such that each d-polytope with $d+1+\alpha$ vertices has $\text{xc} \leq d+\alpha$, it would provide non-trivial upper bounds for xc with respect to the number of vertices...

Generic polytopes

Theorem (P. 2016)

A generic d-polytope P with $d+1+\alpha$ facets has extension complexity

$$\mathrm{xc}(P) \geq 2\sqrt{d(n-1)} - d + 1$$



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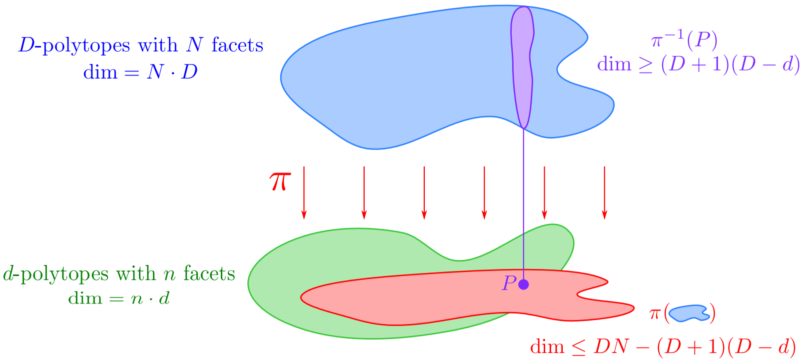

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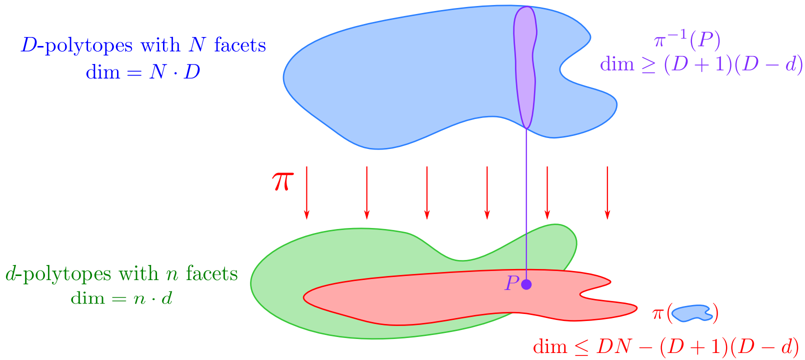

$\Rightarrow$ A generic P has $\mathrm{xc}(P) \geq \min_D \frac{(D+1)(D-d)+nd}{D} = 2\sqrt{nd-d} - d + 1$

Generic polytopes with few vertices

Corollary

For any $d > \left(\frac{\alpha-1}{2}\right)^2$ there are d-polytopes with $d+1+\alpha$ vertices/facets with extension complexity $d+1+\alpha$

Theorem (P. 2016)

Let P be a d-polytope with $d+4$ vertices, then

1. $\mathrm{xc}(P) = d+2$ *if and only if P has $d+2$ facets.*
2. $\mathrm{xc}(P) = d+3$ *if and only if:*
   - 2.1 *P has $d+3$ facets, or*
   - 2.2 *$P = \pi(Q)$, where $Q \cong \mathrm{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ for some affine projection π. In this case, either*
     - 2.2.1 *$P = \mathrm{pyr}_k(Q)$ where Q is a Desarguian hexagon (a hexagon with $\mathrm{xc}(Q) = 5$), or*
     - 2.2.2 *P has a subset of 6 vertices forming a triangular prism.*
3. $\mathrm{xc}(P) = d+4$ *otherwise.*

d-polytopes with $d+4$ vertices

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   Combinatorial condition, 1 combinatorial type
2. $\mathrm{xc}(P) = d+3$ *if and only if:*
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   Combinatorial condition, ≤ 8 combinatorial types
   2.2 $P = \pi(Q)$, *where* $Q \cong \mathrm{pyr}_{d-2}(\Delta_1 \times \Delta_2)$ *for some affine projection π.*
   In this case, either
      2.2.1 $P = \mathrm{pyr}_k(Q)$ *where Q is a Desarguian hexagon (a hexagon with $\mathrm{xc}(Q) = 5$), or*
      Geometric condition, 1 combinatorial type
      2.2.2 *P has a subset of 6 vertices forming a triangular prism.*
      Geometric condition, $\theta(d^2)$ combinatorial types
3. $\mathrm{xc}(P) = d+4$ *otherwise.*
   Superexponentially many, $\geq d^{d(1/2-o(1))}$ combinatorial types

There are few polytopes with few vertices and few facets

Theorem (P. 2016)

$\exists$ $D(\alpha, \beta)$ *such that every d-polytope with* $\leq d + 1 + \alpha$ *vertices,* $\leq d + 1 + \alpha$ *facets and* $d > D(\alpha, \beta)$ *is a pyramid.*

There are few polytopes with few vertices and few facets

Theorem (P. 2016)

$\exists\ D(\alpha,\beta)$ such that every d-polytope with $\leq d+1+\alpha$ vertices, $\leq d+1+\alpha$ facets and $d > D(\alpha,\beta)$ is a pyramid.

Corollary

For fixed α,β, the number of combinatorial types of d-polytopes with $\leq d+1+\alpha$ vertices and $\leq d+1+\alpha$ facets is bounded by a constant independent of d.

There are few polytopes with few vertices and few facets

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Proof.

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Proof.

Theorem (Marcus 1981)

$\exists$ $f(\alpha)$ *such that for* $d > f(\alpha)$ *every d-polytope with* $\leq d + 1 + \alpha$ *vertices has a vertex adjacent to all the other vertices.*

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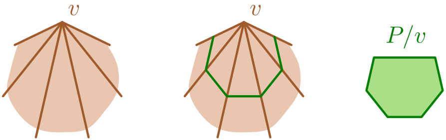
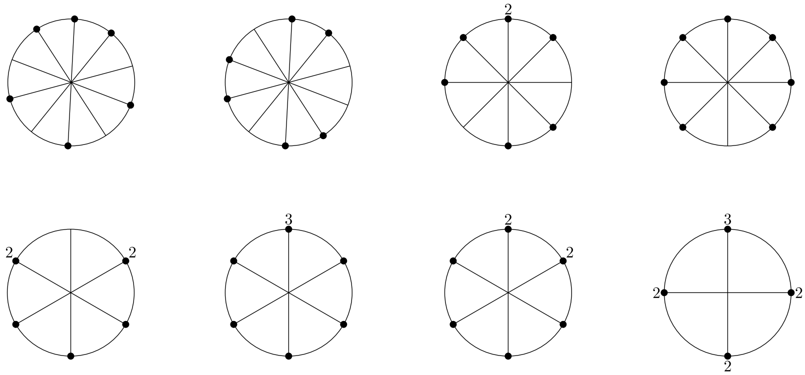
By induction on β. Since $d > D(\alpha,\beta) \geq f(\alpha)$, $\exists v$ adjacent to all the other vertices.



If there is only one facet non-adjacent to $v \Rightarrow P$ is a pyramid.
Otherwise, the *vertex-figure* P/v has $(d-1)+\alpha+1$ vertices and $< (d-1)+\beta+1$ facets. By induction, P/v is a pyramid $\Rightarrow P$ is a pyramid. □

Non-pyramidal d-polytopes with d+4 vertices and d+3 facets



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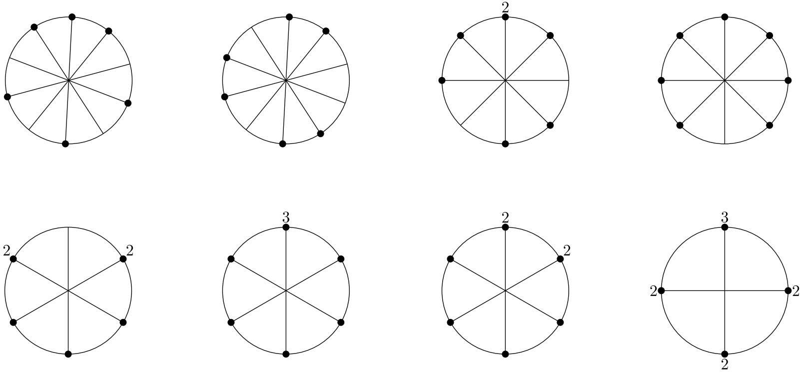

Merci beaucoup!