Algorithm design for topologically structured graphs

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Journées nationales 2017 GDR Informatique Mathématique

Montpellier, du 14 au 16 mars 2017

- Structural Graph Theory, Decompositions, Algorithmic Graph Minors

- Flatness, Disjoint Paths Problem, Irrelevant vertex technique

Prehistory of **structural graph theory**

Theorem

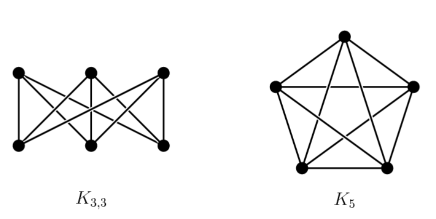
Every planar graph can be properly colored with 4 colors.

Proofs:

[Appel & Haken, Illinois J. of Math., 1977]

[Robertson, Sanders, Seymour, Thomas, JCTSB'97]

Prehistory of **structural graph theory**



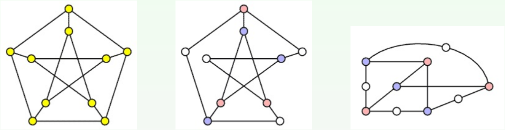
Theorem (Kuratowski)

A graph is planar iff it excludes K_5 and $K_{3,3}$ as topological minors.

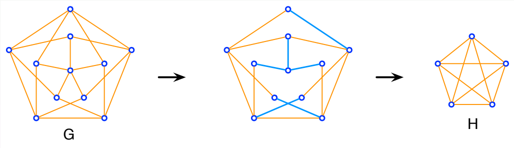
Proof:

[Kazimierz Kuratowski, Fund. Math. 1930]

Topological minor:Remove vertices/edges + dissolve 2-degree vertices



Minor:Remove vertices/edges + contract edges



Alternative statement:

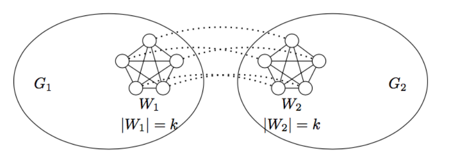
Theorem (Kuratowski & Wagner)

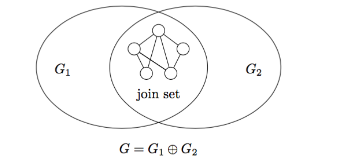
A graph is planar iff it excludes K_5 and $K_{3,3}$ as minors.

Klaus Wagner thought about proving the 4-colour theorem for graphs excluding K_5 as a minor (these graphs contain planar graphs).

Question: How K_5-minor free graphs look like?

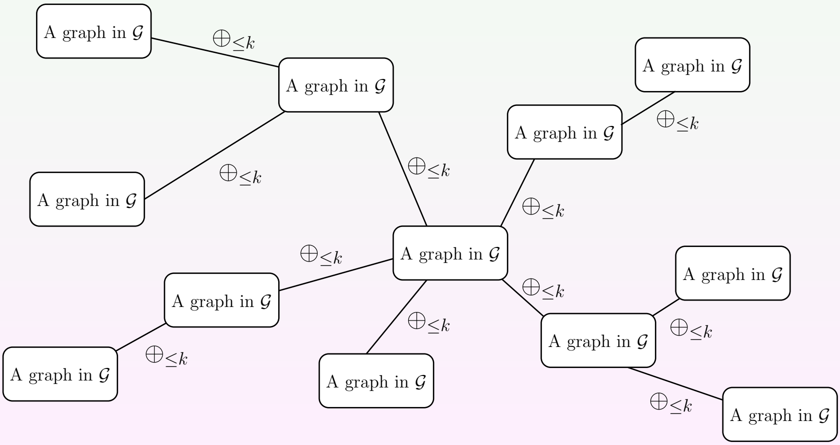
Clique sum operation:



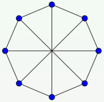


Let $\mathcal{G}$ be a graph class.

A graph G has a *tree decomposition* on $\mathcal{G}$ with adhesion $\leq k$ if it can be constructed by repetitively applying $\leq k$-clique sums on graphs in $\mathcal{G}$.

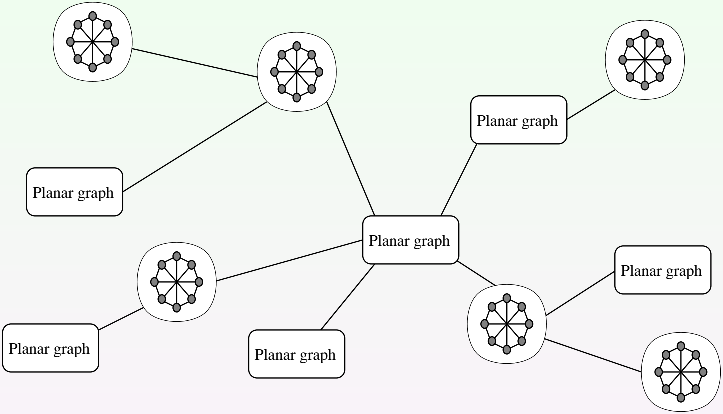


Let $\mathcal{P} = \text{planar graphs}$,

Wagner graph W_8: 

Theorem (Wagner)

K_5-minor free graphs are exactly those that have tree decompositions on $\mathcal{P} \cup \{W_8\}$ with adhesion ≤ 3.



Theorem (Wagner)

K_5-minor free graphs are exactly those that have tree decompositions on $\mathcal{P} \cup \{W_8\}$ with adhesion ≤ 3.

Remarks:

- K_5-minor free graphs are builded by gluing together planar graphs and W_8.
- Some of the properties of planar graphs also hold for K_5-minor free graphs.
- Some algorithms that work for planar graphs also work for K_5-minor free graphs.

Question: What about K_h-minor free graphs for $h \geq 6$?

- **This is a much more difficult question!**

An answer: by Robertson and Seymour in their Graph Minors **GM** series:

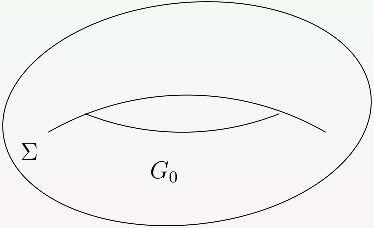

Theorem (GM main structural theorem)

For every h, there is an r such that K_h-minor free graphs have tree decompositions on A_r with adhesion $\leq r$.

A_r= "r-almost embedded graphs with apices"

... we next explain what is this!

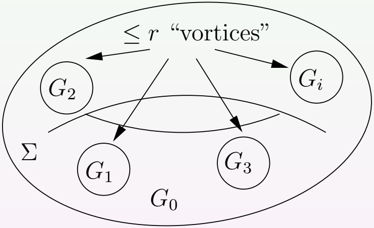
Planar graphs $\rightarrow$ embedded graphs



G_0: embeded in a surface Σ of Euler genus $\leq r$

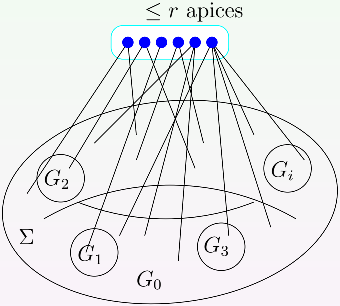
embedded graphs $\rightarrow$ r-almost embedded graphs

r-almost embedded graph $G = G_0 \cup G_1, \ldots, G_i$

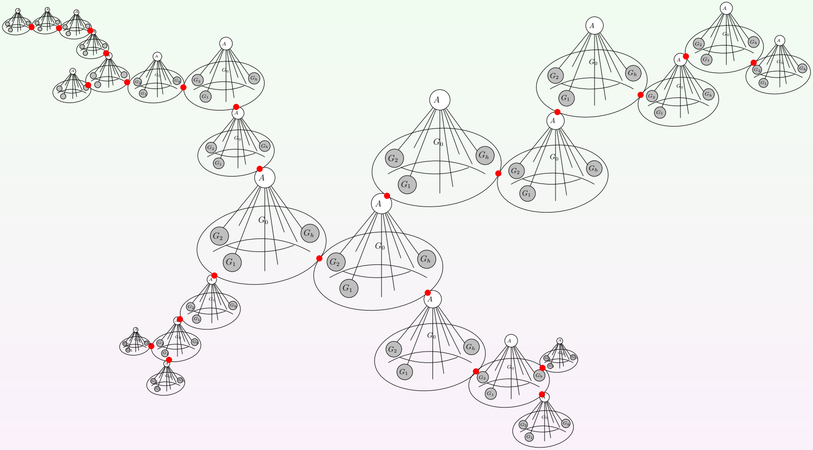


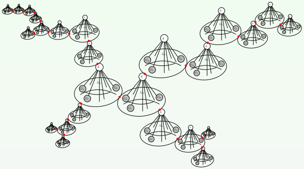
G_0: embeded in a surface Σ of Euler genus $\leq r$

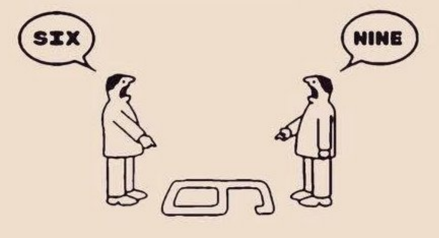
r-almost embedded graphs $\rightarrow$ r-almost embedded graphs with apices



$+$ a set of $\leq r$ vertices (apices)
adjacent to any other vertex


A
G_0
G_1
G_2
G_h





Two structural elements:
How minor-free graphs look like?

1D. Trees

2D. Surfaces

- **Algorithms?**
- (Very) **general idea:**

Combine algorithmic results on graph families 1, 2 above.

- Algorithmic area: Algorithmic Graph Minors

Treewidth

Let $\mathcal{G}_k$ be the set of all graphs on k vertices.

The **treewidth** of a graph G is the minimum k such that G has a tree decomposition on $\mathcal{G}_{k+1}$ with adhesion at most k.

▶ A graph of bounded treewidth can be seen as the result of gluing together bounded size graphs.

▶ treewidth is important in combinatorics (**GM** series)

▶ treewidth is also important in **graph algorithms**

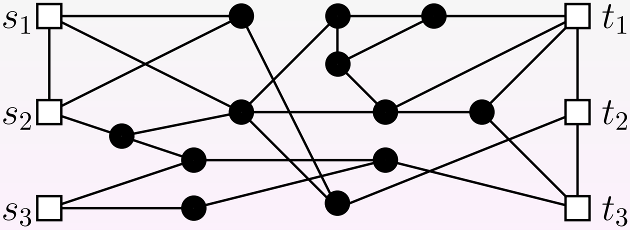
Theorem ([Courcelle], [Seese], & [Borie, Parker & Tovey])

If Π is problem on graphs that is expressible in MSOL then it can be solved in $f(k) \cdot n$ steps where $k = \mathbf{tw}(G)$.

Disjoint Paths

Instance: A graph G and k pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of terminals in G.

Question: Are there k pairwise *vertex disjoint* paths $P_1, \ldots, P_k$ in G such that each P_i has endpoints s_i and t_i?



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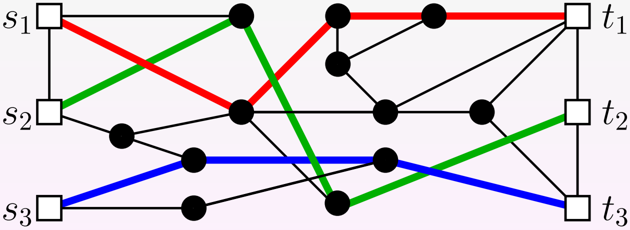


k-Disjoint Paths Problem where solved using the

The irrelevant vertex Technique

introduced in [Roberston, Seymour, JCTSB 1995, **GM** XIII]

The irrelevant vertex

Given an instance (G, T, k) of the k-Disjoint Paths problem, a vertex $v \in V(G)$ is an *irrelevant* vertex of G if (G, T, k) and $(G \setminus v, T, k)$ are equivalent instances of the problem.

Idea:▶ Find irrelevant vertex and recurse!

We give a outline of the idea for Planar Disjoint Paths.

We distinguish two cases:

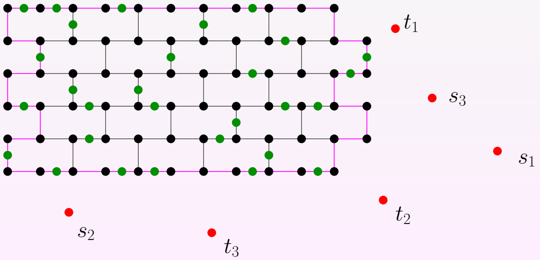
A. The graph has small treewidth [looks like a 1D tree!]

B. The graph has big treewidth [looks like a 2D expanded area!]

Theorem

There is a constant c such that if G is planar and $\mathbf{tw}(G) \geq c \cdot k$ then G contains as a subgraph a subdivision of a k-wall.

a subdivision of a 5-wall

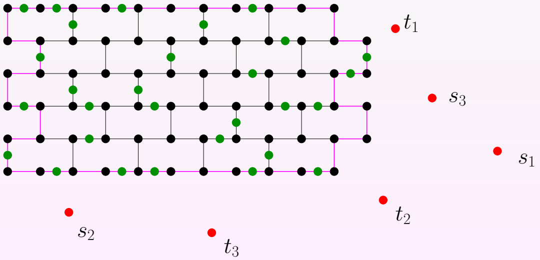


- In the 1D case we use dynamic programming on bounded treewidth graphs (because the problem can be expressed in MSOL).
- In the 2D case we

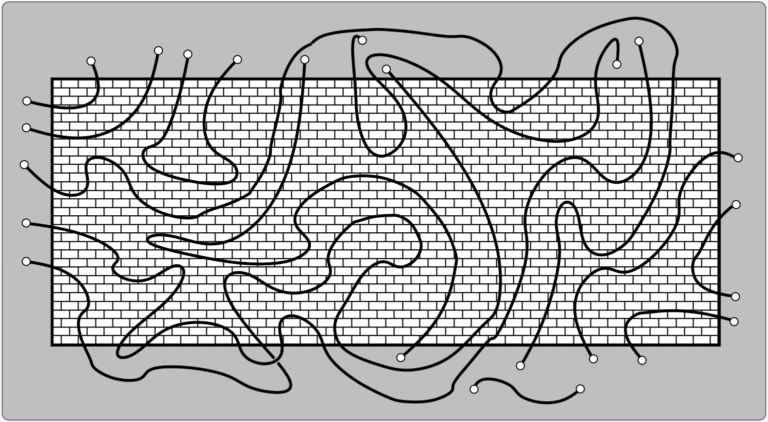
detect a "big enough" subdivided wall W

(2) we consider a "big enough" subdivided subwall W' that does not contain any terminal

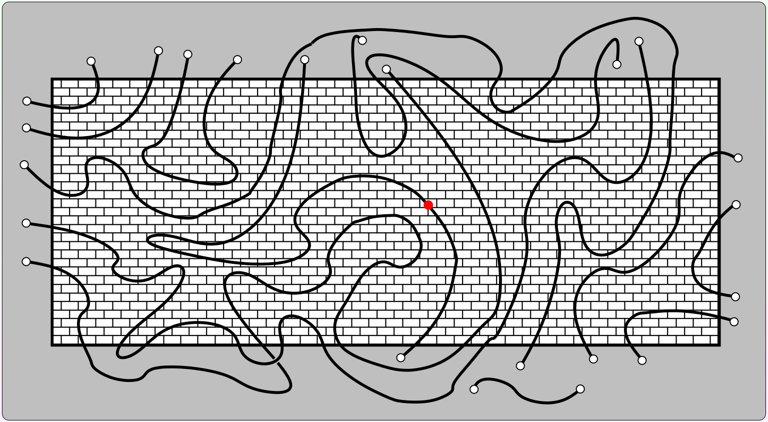
(3) we declare the "middle vertex" of W' irrelevant!



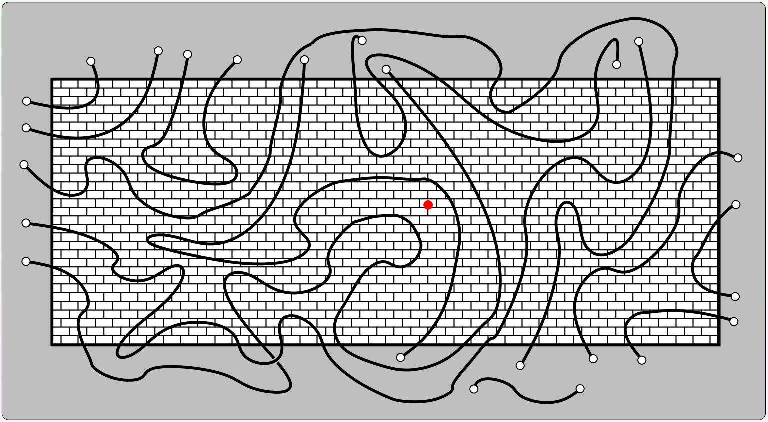
A solution to the k-Disjoint Paths Problem, for $k = 12$



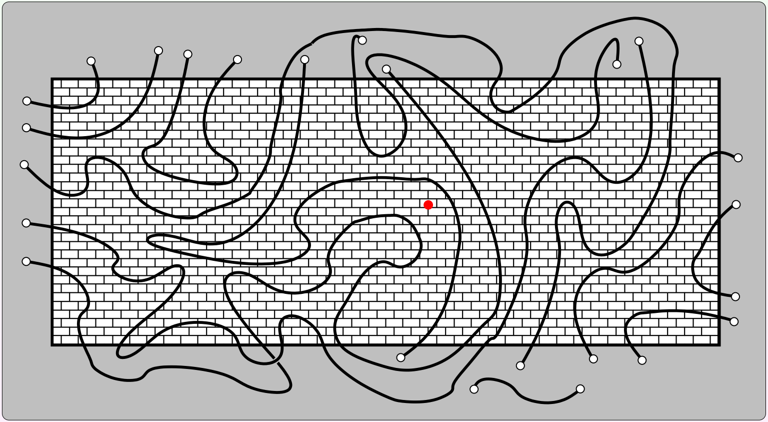
The middle vertex of the subdivided wall W'



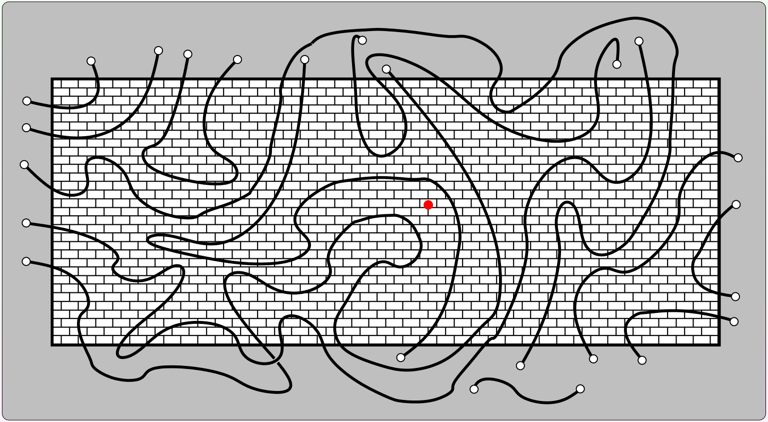
A way to avoid the middle vertex



Is it <u>always</u> possible to avoid the middle vertex?



Answer: YES if the height of W' is "big enough" (function of k)



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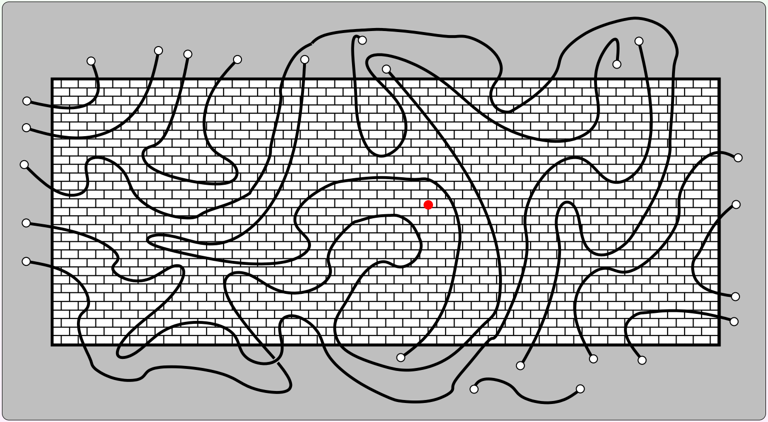


This “big enough”, in the planar case is c^{k}
[Adler, Kolliopoulos, Lokshtanov, Saurabh, Thilikos, JCTSB 2017]

So, at some point the resulting graph will have small treewidth $= 2^{O(k)}$ and the 1D case will apply.

This gives an algorithm running in $f(k) \cdot n^2$ steps.
Here $f(k) = 2^{2^{O(k)}}$

- What to do with DISJOINT PATHS in general?
- Does the structural theorem provide the previous dichotomy between 1D and 2D case?
- The case where G contains a K_h-minor an irrelevant vertex can again be found (omitted here).
- It appears that when G is K_h-minor free, a 1D-2D dichotomy is possible!

Theorem (Weak Structure Theorem)

There exists a recursive function $g : \mathbb{N} \to \mathbb{N}$, such that for every K_h-minor free graph G and one of the following holds:

1. $\textbf{tw}(G) \leq g(h) \cdot k$
2. $\exists X \subseteq V(G)$ *with* $|X| \leq h - 5$ *such that* $G \setminus X$ *contains as a subgraph a flat subdivided wall* W *where* W *has height* k *and the compass of* W *has a rural division* $\mathcal{D}$ *such that each internal flap of* $\mathcal{D}$ *has treewidth at most* $g(k)$.
   (irrelevant vertex is wanted here...!)

[Roberston, Seymour, JCTSB 1995, **GM** XIII]
Recent optimizations:
[Giannopoulou, Thilikos, SIDMA 2013]
[Kawarabayashi, Thomas, Wollan, JCTSB 2017]

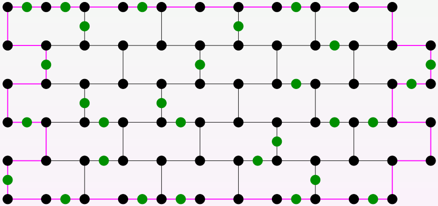
Theorem (Weak Structure Theorem – populist version!)

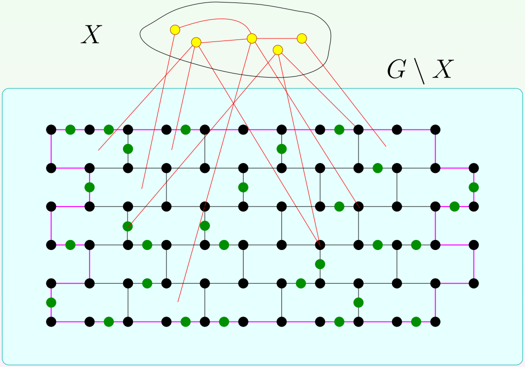
There exists a recursive function $g : \mathbb{N} \to \mathbb{N}$, such that for every K_h-minor free graph G and one of the following holds:

1. $\textbf{tw}(G) \leq g(h) \cdot k$
2. $\exists X \subseteq V(G)$ *with* $|X| \leq h - 5$ *such that* $G \setminus X$ *contains as a subgraph a flat 2D area where 1D trees are planted on it?*
   (irrelevant vertex is still wanted here...!)

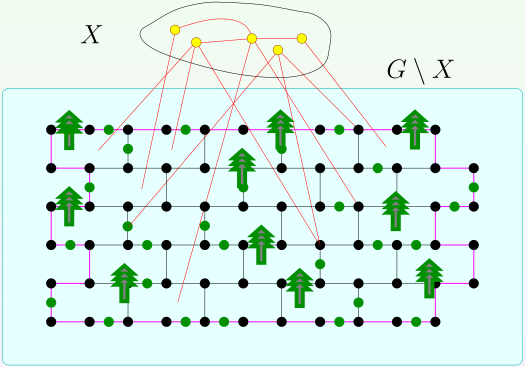
Weak structure theorem $\longrightarrow$ sunny forest theorem!

A subdivided Wall W of heigh 5:



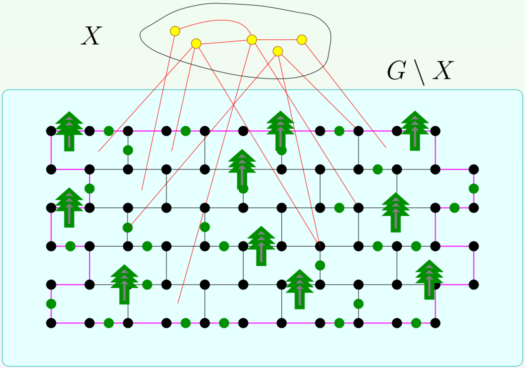


The compass is the part of the $G \setminus X$ that is "inside" the perimeter of the subdivided wall W. The perimeter is as a separator between the internal compass vertices and the part of $G \setminus X$ that is outside the perimeter



The compass can be decomposed to graphs of bounded treewidth (flaps) whose "roots" have size ≤ 3 and form a planar hypergraph inside the disk bounded by the perimeter

Weak structure theorem $\longrightarrow$ sunny forest theorem!



As before, if the height of this subdivided forest is "big enough" we can declare its middle vertex irrelevant!

Now the "big enough" is much worst than the (single exponential) planar case!

Current bounds: bigger than $2^{2^{2^{\Omega(k)}}}$ [Kawarabayashi, Wollan STOC 2010]

- The irrelevant vertex technique is very powerful:

- Its application for each particular problem requires its treatment inside some "*flat*" area of the input graph and the location of some "irrelevant territory" in it.

We next present problems where this machinery has been applied.

Minor Checking:

Check, given an n-vertex graph G and a k-vertex graph H, whether H is a minor of G.

- Solved in $f(k) \cdot n^3$ steps

[Robertson, Seymour, JCTSB 2012]

TOPOLOGICAL MINOR CHECKING:

Check, given an n-vertex graph G and a k-vertex graph H, whether H is a topological minor of G.

- Solved in $f(k) \cdot n^3$ steps

[Grohe, Kawarabayashi, Marx, Wollan:, STOC 2010]

CONTRACTION CHECKING:

Check, given an n-vertex graph G and a k-vertex graph H, whether H is a contraction of G.

► Solved in $f(g, k) \cdot n^3$ steps in graphs of Euler genus g (NP-hard in general even for fixed H's)

[Kamiński, Thilikos, STACS 2012]

VERTEX DISJOINT ODD CYCLES:

Check whether a graph G contains k vertex disjoint odd cycles

- Solved in $f(k) \cdot n^{O(1)}$ steps

[Kawarabayashi, Reed, STOC 2010]

INDUCED DISJOINT PATHS:

same as DISJOINT PATHS but the paths are induced.

- Solved in $f(k) \cdot n$ steps for the planar case

[Kawarabayashi, Kobayashi, JCSS 2012]

CYCLABILITY:

Given G, $R \subseteq V(G)$ and k, check whether every $\leq k$ vertices from R belong into a cycle of G.

- Solved in $2^{2^{O(k)}} \cdot n$ steps for the planar case

(co-W$[1]$-hard in general graphs – no $f(k) \cdot n^{O(1)}$ is expected to exist in general graphs)

[Golovach, Kamiński, Maniatis, Thilikos, SIDMA 2017]

- Can be extended to H-minor free graphs [work in progress]

All the results we explained so far were based on the structural characterisation of K_h-minor free graphs.

Other structural theorems:

- For K_h-topological minor free graphs [Grohe & Marx, SIAM J. Comp.'15]:
- For K_h-immersion free graphs [Wollan JCTSB 2015]:
- ... and more!

Applications: too many to include here!

Research directions

- Extensions to other structures: directed graphs, hypergraphs, matroids, tournaments.
- Optimize parameteric dependencies (or prove lower bounds).
- Project: A meta-algorithmic theory of the irrelevant vertex technque (links with Logic and Automata Theory)
- Other extensions of flatness? (topological vs geometrical)
- Consequences to various algorithmic paradigms: parameterized algorithms, kernelization, approximation algorithms, randomization, counting algorithms, distributed algorithms.

Merci beaucoup

Woman Asleep at a Table, 1936 – Pablo Picasso
(Malaga, Andalousie, Espagne 1881-1973, Mougins, France)